Université de Lille

École doctorale **ED MADIS - 631**

Unité de recherche **Laboratoire Paul Painlevé - UMR 8524**

Thèse présentée par **Marvin Verstraete**

Soutenue le **21 octobre 2025**

En vue de l'obtention du grade de docteur de l'Université de Lille

Discipline **Mathématiques**

Équations de Maurer-Cartan et homotopie des espaces d'applications opéradiques en caractéristique positive

Thèse dirigée par Benoit Fresse

Composition du jury

Rapporteurs	Fernando Muro	Professor, Universidad de Sevilla	
	Geoffroy Horel	Maître de conférences, Université Sorbonne Paris Nord	
Examinateurs	Antoine Touzé	Professeur, Université de Lille	président du jury
	Christopher Rogers	Associate professor, University of Nevada	
	Muriel Livernet	Professeure, Université de Paris Cité	
Directeur de thèse	Benoit Fresse	Professeur, Université de Lille	

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Doctoral School **ED MADIS - 631**

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Thesis defended by **Marvin Verstraete**

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Academic Field **Mathematics**

Maurer-Cartan equations and homotopy of operadic mapping spaces in positive characteristic

Thesis supervised by Benoit FRESSE

Committee members

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	Geoffroy HOREL	Maître de conférences, Université Sorbonne Paris Nord	
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À mon oncle Jean-Pierre

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Équations de Maurer-Cartan et homotopie des espaces d'applications opéradiques en caractéristique positive

Résumé

Le but de cette thèse est d'étudier le type d'homotopie d'espaces d'applications opéradiques. Dans une première partie, nous étudions les algèbres pré-Lie différentielles graduées à puissances divisées sur un anneau de caractéristique positive, et construisons une théorie de la déformation contrôlée par ces algèbres. Nous montrons que cette théorie de la déformation admet un analogue au théorème de Goldman-Millson, puis appliquons cette théorie au calcul des composantes connexes d'un espace d'applications dans la catégorie des opérades symétriques. Dans une seconde partie, nous définissons et étudions une notion d'espace de Maurer-Cartan simplicial associé à une algèbre brace complète. Nous montrons qu'une algèbre brace détermine une algèbre pré-Lie à homotopie près simpliciale à puissances divisées. L'espace de Maurer-Cartan simplicial est donné par les solutions de l'équation de Maurer-Cartan dans cette algèbre. Nous déterminons le type d'homotopie de cet ensemble simplicial, et montrons une extension du théorème de Goldman-Millson satisfaite par ces espaces simpliciaux. Nous montrons que nous pouvons décrire un espace d'applications dans la catégorie des opérades non symétriques en tant qu'espace de Maurer-Cartan simplicial associé à une certaine brace algèbre complète. Nous exprimons enfin un espace d'application dans la catégorie des opérades symétriques en tant qu'ensemble de Maurer-Cartan degré par degré de certaines algèbres pré-Lie à homotopie près à puissances divisées.

Mots clés : opérades, théorie de la déformation, algèbres pré-lie, topologie algébrique

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Maurer-Cartan equations and homotopy of operadic mapping spaces in positive characteristic

Abstract

The aim of this thesis is to study the homotopy of operadic mapping spaces. In the first part, we study differential graded pre-Lie algebras with divided powers and construct a deformation theory controled by these algebras. We prove that this deformation theory admits an analogue of the Goldman-Millson theorem, and apply this theory to the computation of the connected components of a mapping space in the category of symmetric operads. In a second part, we define and study a notion of simplicial Maurer-Cartan set associated to a complete brace algebra. For this purpose, we use that any brace algebra determines a simplicial pre-Lie algebra up to homotopy with divided powers. The simplicial Maurer-Cartan set is given by the solutions of the Maurer-Cartan equation in this algebra. We determine the homotopy type of this simplicial Maurer-Cartan set and prove an extension of the Goldman-Millson theorem. We show that we can describe a mapping space in the category of non-symmmetric operads as a simplicial Maurer-Cartan set of some complete brace algebra. We finally describe a mapping space in the category of symmetric operads as a degree-wise Maurer-Cartan set of some pre-Lie algebras up to homotopy with divided powers.

Keywords: operads, deformation theory, pre-lie algebras, algebraic topology

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Introduction (en français)

Dans cette thèse, nous étudions l'homotopie des espaces d'applications associés à des opérades définies sur un corps de caractéristique positive.

Opérades et algèbres sur une opérade

La notion d'opérade a été introduite afin de définir des catégories d'algèbres décrites par des opérations. Cette notion a initialement été introduite dans l'étude des espaces de lacets (voir [May06]). Les opérades sont depuis utilisées dans de nombreux domaines des mathématiques.

Pour notre étude, nous considérons principalement des opérades dans une catégorie des $\mathbb{K}$-modules gradués et munis d'une différentielle (ou dg $\mathbb{K}$-modules). Les exemples fondamentaux d'opérades sont définies sur une catégorie de modules. Nous pouvons voir ces opérades comme des opérades dans les dg $\mathbb{K}$-modules concentrés en degré 0. Nous travaillons dans la catégorie des dg $\mathbb{K}$-modules afin de faire de la théorie de l'homologie et de donner des applications en théorie de l'homotopie.

Dans cette configuration, une opérade est une suite de dg $\mathbb{K}$-modules $(\mathcal{P}(n))_n$ telle que chaque $\mathcal{P}(n)$ est muni d'une action du groupe symétrique Σ_n, et nous avons des opérations de composition partielles

$$\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \longrightarrow \mathcal{P}(n+m-1)$$

pour tout $1 \leq i \leq n$, qui satisfont des axiomes d'associativité, d'unité et d'équivariance. Pour tout $p \in \mathcal{P}(n)$ et $q \in \mathcal{P}(m)$, l'opération $p \circ_i q$ représente l'insertion de l'opération q à la i-ème variable de l'opération p. Un morphisme d'opérades $\mathcal{P} \longrightarrow \mathcal{Q}$ est une suite de morphismes $\mathcal{P}(n) \longrightarrow \mathcal{Q}(n)$ qui préservent les compositions opéradiques de $\mathcal{P}$ et $\mathcal{Q}$ et l'action des groupes symétriques.

Le premier exemple d'opérade est l'opérade des endomorphismes End_V associé à un dg $\mathbb{K}$-module V. Cette opérade est définie par $\mathrm{End}_V(n) = \mathrm{Hom}(V^{\otimes n}, V)$ pour tout $n \geq 0$, où la i-ème composition de $f \in \mathrm{End}_V(p)$ par $g \in \mathrm{End}_V(q)$ est définie par

$$(f \circ_i g)(v_1 \otimes \cdots \otimes v_{p+q-1}) = \pm f(v_1 \otimes \cdots \otimes v_{i-1} \otimes g(v_i \otimes \cdots \otimes v_{i+q-1}) \otimes v_{i+q} \otimes \cdots \otimes v_{p+q-1})$$

pour tout $v_1, \ldots, v_{p+q-1} \in V$.

Pour $\mathcal{P}$ une opérade, une $\mathcal{P}$-algèbre est un dg $\mathbb{K}$-module A muni de morphismes

$$\mathcal{P}(n) \otimes A^{\otimes n} \longrightarrow A$$

compatibles avec la composition opéradique de $\mathcal{P}$. Les éléments de $p \in \mathcal{P}(n)$ sont donc vus comme des applications de $A^{\otimes n}$ vers A. Se donner une structure de $\mathcal{P}$-algèbre sur A revient ainsi à se donner un morphisme d'opérades $\mathcal{P} \longrightarrow \mathrm{End}_A$.

Ce formalisme permet de retrouver de nombreuses structures algébriques classiques. Par exemple, il existe une opérade $\mathcal{C}om$ qui gouverne les algèbres associatives et commutatives, une opérade $\mathcal{A}s$ qui gouverne les algèbres associatives, une opérade $\mathcal{L}ie$ qui gouverne les algèbres de Lie...

Dans cette thèse, nous aurons souvent besoin de donner un sens à des sommes infinies pouvant notamment apparaître dans certaines équations de Maurer-Cartan. Nous utiliserons pour cela une notion de $\mathcal{P}$-algèbre complète que nous formons dans une catégorie de dg $\mathbb{K}$-modules filtrés complets. Une filtration sur un dg $\mathbb{K}$-module V est une suite de dg $\mathbb{K}$-modules $(F_nV)_{n\geq 1}$ telle que

$$\cdots \subset F_nV \subset F_{n-1}V \subset \cdots \subset F_1V = V.$$

La complétion de V pour cette filtration est définie par $\widehat{V} := \lim_{n\geq 1} V/F_nV$. Le dg $\mathbb{K}$-module V est dit complet pour sa filtration si $V \simeq \widehat{V}$. En particulier, pour tout dg $\mathbb{K}$-module V muni d'une filtration, la complétion $\widehat{V}$ est complet. Si W est un autre dg $\mathbb{K}$-module, le produit tensoriel $V \otimes W$ au-dessus de $\mathbb{K}$ est aussi muni d'une filtration définie par

$$F_n(V \otimes W) = \bigoplus_{p+q=n} F_pV \otimes F_qW.$$

En général, même si V et W sont complets pour leur filtration, le produit tensoriel $V \otimes W$ n'est pas complet. On définit alors le produit tensoriel complet par $V \widehat{\otimes} W := \widehat{V \otimes W}$, en considérant la filtration sur $V \otimes W$ définie précédemment. Une $\mathcal{P}$-algèbre complète est ainsi une $\mathcal{P}$-algèbre telle que, pour tout $p \in \mathcal{P}(n)$, le morphisme $V^{\otimes n} \longrightarrow V$ induit par p préserve les filtrations de $V^{\otimes n}$ et V.

Espace d'applications simplicial

La notion d'espace simplicial d'applications est définie dans un contexte général. L'idée est de formaliser des propriétés qui reflètent un modèle simplicial des espaces d'applications en topologie. Dans tout ce qui suit, nous utilisons l'expression "espace d'application" pour "espace d'application simplicial" puisque nous utiliserons uniquement cette version simpliciale de la notion d'espace d'application.

La catégorie des espaces topologiques $\mathcal{T}op$ est munie d'un foncteur $\mathrm{Map}_{\mathcal{T}op}(-,-) : \mathcal{T}op^{op} \times \mathcal{T}op \longrightarrow \mathrm{sSet}$ qui confère à $\mathcal{T}op$ une structure de catégorie enrichie sur la catégorie des ensembles simpliciaux. Ce foncteur peut être utilisé pour encoder des notions d'homotopie supérieures dans la catégorie $\mathcal{T}op$. Pour tout espaces topologiques X, Y, les composantes connexes de $\mathrm{Map}_{\mathcal{T}op}(X, Y)$ sont en bijection avec les classes de morphismes $X \longrightarrow Y$ pour la relation d'homotopie, tandis que les groupes d'homotopies encodent des relations d'homotopie supérieures. Cette approche nous permet d'utiliser des outils issues de la topologie algébrique afin d'étudier les morphismes à homotopie près dans $\mathcal{T}op$.

L'ensemble simplicial $\mathrm{Map}_{\mathcal{T}op}(X,Y)$ est défini de la façon suivante. Pour tout $X \in \mathcal{T}op$, on définit deux foncteurs $X \otimes - : \mathrm{sSet} \longrightarrow \mathcal{T}op$ et $X^{-} : \mathrm{sSet}^{op} \longrightarrow \mathcal{T}op$ par

$$X \otimes K := X \times |K| \quad ; \quad X^K := \mathrm{Mor}_{\mathcal{T}op}(|K|, X),$$

pour tout $K \in \mathrm{sSet}$, où $|K|$ désigne la réalisation géométrique de l'ensemble simplicial K. Pour tout $X, Y \in \mathcal{T}op$ et $K \in \mathrm{sSet}$, nous avons l'isomorphisme

$$\mathrm{Mor}_{\mathcal{T}op}(X \otimes K, Y) \simeq \mathrm{Mor}_{\mathcal{T}op}(X, Y^K).$$

On pose alors $\mathrm{Map}_{\mathcal{T}op}(X,Y) := \mathrm{Mor}_{\mathcal{T}op}(X \otimes \Delta^\bullet, Y)$, où, pour tout $n \geq 0$, nous désignons par Δ^n le n-simplexe fondamental.

Dans une catégorie de modèle C quelconque, il est possible de généraliser partiellement ces résultats. Pour tout objets $X, Y \in C$, il existe un objet cosimplicial $X \otimes \Delta^\bullet$ appelé repère cosimplicial associé à X et un objet simplicial $Y^{\Delta^\bullet}$ appelé repère simplicial associé à Y tels que les deux ensembles simpliciaux $\mathrm{Mor}_C(X \otimes \Delta^\bullet, Y)$ et $\mathrm{Mor}_C(X, Y^{\Delta^\bullet})$ soient des complexes de Kan reliés par un zig-zag d'équivalences faibles. On peut définir un ensemble simplicial par $\mathrm{Map}_C(X,Y) = \mathrm{Mor}_C(X \otimes \Delta^\bullet, Y)$, ou de façon équivalente par $\mathrm{Map}_C(X,Y) = \mathrm{Mor}_C(X, Y^{\Delta^\bullet})$, de sorte qu'on ait à nouveau une bijection entre ses composantes connexes et les classes d'homotopie de morphismes $X \longrightarrow Y$.

Nous nous intéressons au cas où C est la catégorie des opérades non symétriques, ou la catégorie des opérades symétriques connexes. Dans ce cas, l'objet simplicial $\mathrm{Map}_{\mathcal{O}p}(\mathcal{Q}, \mathcal{P})$ permet une compréhension fine des morphismes d'opérades de $\mathcal{Q}$ vers $\mathcal{P}$ à homotopie près. La principale motivation pour l'étude de ces objets dans la catégorie des opérades est que si $\mathcal{P} = \mathrm{End}_V$, alors l'espace $\mathrm{Map}_{\mathcal{O}p}(\mathcal{Q}, \mathrm{End}_V)$ détermine l'homotopie d'un espace de module des structures de $\mathcal{Q}$-algèbres sur V.

Etat de l'art en caractéristique nulle

Soit $\mathcal{Q} = B^c(\mathcal{C})$ la construction cobar d'une coopérade $\mathcal{C}$ coaugmentée telle que $\mathcal{C}(0) = 0$. L'étude du type d'homotopie de $\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P})$ est déjà connue dans le cas où $\mathbb{K}$ est un corps de caractéristique nulle. Le calcul des groupes d'homotopie a notamment été effectué dans [Yal16], dans le cas propéradique, en utilisant un repère simplicial explicite associé à $\mathcal{P}$. Ce repère simplicial explicite est défini par

$$\mathcal{P}^{\Delta^\bullet} := \mathcal{P} \otimes \Omega^*(\Delta^\bullet)$$

où, pour tout $n \geq 0$, le dg $\mathbb{K}$-module $\Omega^*(\Delta^n)$ désigne l'algèbre de Sullivan des formes polynomiales de de Rham sur Δ^n (voir par exemple [BG76, §2.1]). Nous obtenons alors

$$\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P}) = \mathrm{Mor}_{\Sigma\mathcal{O}p}(\mathcal{Q}, \mathcal{P}^{\Delta^\bullet}).$$

Les n-simplexes de $\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P})$ sont donc identifiés à des éléments de $\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})\widehat{\otimes}\Omega^*(\Delta^n)$ satisfaisant certaines équations, où $\overline{\mathcal{C}}$ est le coidéal de coaugmentation de $\mathcal{C}$ et $\overline{\mathcal{P}}$ l'idéal d'augmentation de $\mathcal{P}$.

Ces équations, appelées équations de Maurer-Cartan, peuvent être décrites en utilisant une structure d'algèbre de Lie sur $\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}})\widehat{\otimes}\Omega^*(\Delta^n)$. Cette structure d'algèbre de Lie peut être déduite par une structure d'algèbre pré-Lie sur $\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}})$. Voici les détails de cette construction.

Rappelons que si L est une algèbre de Lie complète, alors un élément de Maurer-Cartan est un élément $\tau \in L_{-1}$ tel que

$$d(\tau) + \frac{1}{2}[\tau,\tau] = 0.$$

On note $\mathcal{MC}(L)$ l'ensemble des éléments de Maurer-Cartan de L. Tout élément de Maurer-Cartan $\tau \in \mathcal{MC}(L)$ induit une différentielle d_τ sur L définie par

$$d_\tau(x) = d(x) + [\tau, x]$$

pour tout $x \in L$. On désigne par L^τ le dg $\mathbb{K}$-module L muni de la différentielle d_τ. En utilisant la structure d'algèbre commutative de $\Omega^*(\Delta^n)$ pour tout $n \geq 0$, on peut munir le dg $\mathbb{K}$-module $L\widehat{\otimes}\Omega^*(\Delta^n)$ d'une structure d'algèbre de Lie complète. On définit alors l'ensemble de Maurer-Cartan simplicial associé à L par

$$\mathcal{MC}_\bullet(L) = \mathcal{MC}(L\widehat{\otimes}\Omega^*(\Delta^\bullet)).$$

Les algèbres pré-Lie sont des cas particuliers d'algèbres de Lie. Une algèbre pré-Lie est un dg $\mathbb{K}$-module L munie d'une opération $\star : L \otimes L \longrightarrow L$ telle que

$$(x \star y) \star z - x \star (y \star z) = (-1)^{|y||z|}((x \star z) \star y - x \star (z \star y))$$

pour tout $x, y, z \in L$. Le crochet de Lie induit par la structure pré-Lie est donné par

$$[x, y] = x \star y - (-1)^{|x||y|} y \star x.$$

La structure d'algèbre pré-Lie de $\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}})$ est donnée par une composition

$$f \star g : \overline{\mathcal{C}} \xrightarrow{\Delta_{(1)}} \overline{\mathcal{C}} \circ_{(1)} \overline{\mathcal{C}} \xrightarrow{\overline{f}\circ_{(1)}\overline{g}} \overline{\mathcal{P}} \circ_{(1)} \overline{\mathcal{P}} \xrightarrow{\gamma_{(1)}} \overline{\mathcal{P}}$$

pour tout $f, g \in \mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}})$, où $\Delta_{(1)}, \circ_{(1)}$ et $\gamma_{(1)}$ désignent les composantes infinitésimales de structures de composition opéradiques et coopéradiques (voir [LV12, §6.4.4]).

On a une bijection entre les morphismes d'opérades $B^c(\mathcal{C}) \longrightarrow \mathcal{P}^{\Delta^n}$ et les éléments de Maurer-Cartan de $\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}})\widehat{\otimes}\Omega^*(\Delta^n)$ pour tout $n \geq 0$, de sorte que

$$\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}),\mathcal{P}) = \mathcal{MC}_\bullet(\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}})).$$

Le calcul des groupes d'homotopie de $\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}),\mathcal{P})$ peut ainsi être effectué via le calcul général des groupes d'homotopies de $\mathcal{MC}_\bullet(L)$ associé à une algèbre de Lie L (voir [Ber15, Theorem 1.1]). Explicitement, pour tout $k \geq 0$ et pour tout $\tau \in \mathcal{MC}(L)$,

$$\pi_{k+1}(\mathcal{MC}_\bullet(L),\tau) \simeq H_k(L^\tau),$$

où $H_0(L^\tau)$ est muni de la structure de groupe donnée par la formule de Baker-Campbell-Hausdorff.
Le calcul des composantes connexes de $\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P})$ peut s'effectuer de la façon suivante. On utilise d'abord l'identification formelle

$$\pi_0 \mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P}) = \mathrm{Mor}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P})/\sim$$

où $\sim$ désigne la relation d'équivalence d'homotopie dans la catégorie de modèles $\Sigma\mathcal{O}p^0$ (voir [Fre17b, Theorem 3.2.14]). Le calcul du membre de droite peut être effectué en utilisant la théorie de la déformation pré-Lie construite dans [DSV16] qui généralise celle des algèbres de Lie. Explicitement, un élément de Maurer-Cartan τ dans une algèbre pré-Lie L est un élément $\tau \in L_{-1}$ tel que

$$d(\tau) + \tau \star \tau = 0.$$

Le groupe de jauge $(L_0, BCH, 0)$ peut également être décrit en termes d'opérations pré-Lie. Considérons le sous-ensemble $1 + L_0 \subset \mathbb{K} \oplus L$. Sous des hypothèses de convergence, on définit le produit circulaire $\circledcirc : L \otimes (1 + L_0) \longrightarrow L$ par

$$x \circledcirc (1 + y) = \sum_{n \geq 0} \frac{1}{n!} x\{\underbrace{y, \ldots, y}_{n}\},$$

pour tout $x \in L$ et $y \in L_0$, où nous désignons par $-\{-, \ldots, -\}$ les opérations brace symétriques dans l'algèbre pré-Lie L (voir [OG08] ou [LM05]). Nous pouvons restreindre ce produit circulaire en une opération sur $1 + L_0$ définie par

$$(1 + x) \circledcirc (1 + y) = 1 + y + \sum_{n \geq 0} \frac{1}{n!} x\{\underbrace{y, \ldots, y}_{n}\}$$

pour tout $x, y \in L_0$. Alors le triplet $(1 + L_0, \circledcirc, 1)$ est un groupe qui est isomorphe au groupe de jauge (voir [DSV16, Théorème 2]). Le groupe $(1 + L_0, \circledcirc, 1)$ agit également sur $\mathcal{MC}(L)$ via la formule

$$(1 + \mu) \cdot \tau = (\tau + \mu \star \tau - d(\mu)) \circledcirc (1 + \mu)^{-1}$$

pour tout $\mu \in L_0$ et $\tau \in \mathcal{MC}(L)$. On définit alors le groupoïde de Deligne, noté $\mathrm{Deligne}(L)$, comme étant la catégorie ayant $\mathcal{MC}(L)$ pour ensemble d'objets, et les éléments de $1 + L_0$ en tant que morphismes.

Rappelons que $B^c(\mathcal{C})$, pour une coopérade coaugmentée $\mathcal{C}$, est l'opérade définie par $B^c(\mathcal{C}) = (\mathcal{F}(\Sigma^{-1}\overline{\mathcal{C}}), \partial)$ où nous appliquons le foncteur des opérades libres $\mathcal{F}$ sur la désuspension du coidéal de coaugmentation de notre coopérade $\mathcal{C}$, et ∂ est une dérivation déterminée par les coproduits de composition partielles de $\mathcal{C}$, qui est ajouté à la différentielle interne de $\mathcal{C}$ afin de produire la différentielle de $B^c(\mathcal{C})$. La relation d'homotopie de morphismes de $B^c(\mathcal{C})$ vers $\mathcal{P}$ est déterminée à l'aide de l'objet cylindre explicite associé à $B^c(\mathcal{C})$ fourni par [Fre17b, Theorem 3.2.14]. On a une dérivation tordue explicite ∂ sur $\mathcal{F}(\Sigma^{-1}\overline{\mathcal{C}} \otimes N_*(\Delta^1))$ telle que

$$B^c(\mathcal{C}) \otimes \Delta^1 := (\mathcal{F}(\Sigma^{-1}\overline{\mathcal{C}} \otimes N_*(\Delta^1)), \partial)$$

est un objet cylindre associé à $B^c(\mathcal{C})$, où $N_*(\Delta^1)$ est le complexe de chaines normalisé associé à l'ensemble simplicial Δ^1. En écrivant la relation de commutation avec la différentielle ∂, on trouve que se donner une homotopie d'un morphisme $f : B^c(\mathcal{C}) \longrightarrow \mathcal{P}$ vers un autre morphisme $g : B^c(\mathcal{C}) \longrightarrow \mathcal{P}$ est équivalent au fait que les éléments de $\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})$ correspondant à f et g sont dans la même orbite sous l'action du groupe de jauge de $\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})$ (voir [DSV16, Corollary 2]). On obtient alors une bijection

$$\pi_0\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P}) \simeq \pi_0\mathrm{Deligne}(\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}}))$$

où le membre de droite désigne l'ensemble des classes d'isomorphismes du groupoïde $\mathrm{Deligne}(\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}}))$.

Obstacles en caractéristique positive et idées de la thèse

Le but de cette thèse est de généraliser cette approche à la caractéristique positive. Plusieurs problèmes apparaissent lorsque nous considérons un anneau $\mathbb{K}$ de caractéristique positive.

Premièrement, tous les calculs précédents (plus particulièrement pour le calcul des composantes connexes) utilise des coefficients rationnels. Nous proposons d'utiliser des (généralisations) de puissances divisées afin de maîtriser les dénominateurs non triviaux qui apparaissent dans les formules. Par exemple, dans le chapitre 1, nous utilisons des structures à puissances divisées sur les algèbres pré-Lie afin d'effectuer le calcul des composantes connexes de $\mathrm{Map}_{\Sigma\mathcal{O}p}(B^c(\mathcal{C}), \mathcal{P})$. Dans le chapitre 2, nous utilisons des structures de puissances divisées sur les algèbres pré-Lie à homotopie près afin d'effectuer le calcul des groupes d'homotopie de $\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}), \mathcal{P})$.

Deuxièmement, le repère simplicial $\mathcal{P}^{\Delta^\bullet} = \mathcal{P} \otimes \Omega^*(\Delta^\bullet)$, que nous utilisons afin de calculer les groupes d'homotopie, n'est plus un repère simplicial lorsque $\mathbb{K}$ est un corps de caractéristique positive. Entre autre, ceci provient du fait que la cohomologie de $\Omega^*(\Delta^\bullet)$ est non nulle lorsque le corps de base est de caractéristique positive. Dans le chapitre 2, dans le cas des opérades non symétriques, nous construisons à la place un repère cosimplicial $B^c(\mathcal{C}) \otimes \Delta^\bullet$ associé à la construction cobar d'une coopérade coaugmentée non symétrique $\mathcal{C}$. Nous utilisons ce repère cosimplicial explicite afin de calculer $\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}), \mathcal{P})$.

Résultats du chapitre 1

La théorie de la déformation développée dans [DSV16] ne permet pas d'identifier $\pi_0\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P})$ avec les classes d'isomorphismes d'un groupoïde de Deligne si $char(\mathbb{K}) > 0$, puisque les formules définissant le groupe de jauge associé à une algèbre pré-Lie L, et son action sur les éléments de Maurer-Cartan, font intervenir des formules avec des coefficients rationnels.

Dans ce premier chapitre, nous développons une théorie de la déformation généralisant la théorie de la déformation contrôlée par des algèbres pré-Lie construite dans [DSV16] sur un corps de caractéristique positive, et l'appliquons au calcul de $\pi_0\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P})$. L'idée est de généraliser cette théorie de la déformation en utilisant des algèbre pré-Lie à puissances divisées.

La notion de $\mathcal{P}$-algèbre à puissances divisées a été introduite par Fresse dans [Fre00], où $\mathcal{P}$ est une opérade telle que $\mathcal{P}(0) = 0$. Se donner une structure de $\mathcal{P}$-algèbre sur V est équivalent à se donner une structure d'algèbre sur la monade $\mathcal{S}(\mathcal{P}, -) : \mathrm{dgMod}_{\mathbb{K}} \longrightarrow \mathrm{dgMod}_{\mathbb{K}}$ appelée foncteur de Schur définie par

$$\mathcal{S}(\mathcal{P}, V) := \bigoplus_{n\geq 0} \mathcal{P}(n) \otimes_{\Sigma_n} V^{\otimes n}$$

où nous considérons l'action diagonale de Σ_n sur $\mathcal{P}(n) \otimes V^{\otimes n}$ donnée par son action sur $\mathcal{P}(n)$ et son action par permutations des facteurs sur $V^{\otimes n}$, et où $\otimes_{\Sigma_n}$ désigne un module de coinvariants pour cette action diagonale. Si $\mathcal{P}(0) = 0$, alors le foncteur analogue donné par des invariants

$$\Gamma(\mathcal{P}, V) := \bigoplus_{n\geq 1} \mathcal{P}(n) \otimes^{\Sigma_n} V^{\otimes n}$$

admet également une structure de monade (voir [Fre00, §1.1.18]). Dans ce cas, on a un morphisme de monades

$$Tr : \mathcal{S}(\mathcal{P}, -) \longrightarrow \Gamma(\mathcal{P}, -)$$

donné par le morphisme trace usuel allant des coinvariants vers les invariants, lorsqu'on considère l'action de Σ_n sur $\mathcal{P}(n) \otimes V^{\otimes n}$ pour tout $n \geq 1$. Une $\mathcal{P}$-algèbre à puissances divisées est un dg $\mathbb{K}$-module muni d'une structure d'algèbre sur la monade $\Gamma(\mathcal{P}, -)$. En particulier, toute $\mathcal{P}$-algèbre à puissances divisées est munie d'une structure de $\mathcal{P}$-algèbre induite par le morphisme Tr.

La notion d'algèbre pré-Lie à puissances divisées (ou $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algèbre) a été étudiée par Césaro dans [Ces18], dans la catégorie des $\mathbb{K}$-modules non gradués. L'auteur montre en particulier que toute algèbre pré-Lie à puissances divisées est munie d'opérations braces à poids $-\{-, \ldots, -\}_{r_1,\ldots,r_n}$, définies pour toute collection d'entiers $r_1, \ldots, r_n \geq 0$, qui satisfont des formules similaires aux opérations

$$x\{y_1, \ldots, y_n\}_{r_1,\ldots,r_n} = \frac{1}{\prod_i r_i!} x\{\underbrace{y_1, \ldots, y_1}_{r_1}, \ldots, \underbrace{y_n, \ldots, y_n}_{r_n}\}$$

dans une algèbre pré-Lie sur un corps de caractéristique nulle (voir [Ces18, Propositions 5.9-5.10] pour une liste précise de ces formules).

Toute dg algèbre pré-Lie à puissances divisées est munie d'opérations brace à poids $-\{-, \ldots, -\}_{r_1,\ldots,r_n}$ similaires qui satisfont des versions graduées des formule satisfaites par les braces à poids dans le cas non gradué. Dans le cas gradué, nous avons un analogue de l'équation de Maurer-Cartan :

$$d(x) + x\{x\}_1 = 0.$$

Sous des hypothèses de convergence, nous définissons le produit circulaire par

$$a \circledcirc (1+b) = \sum_{n\geq 0} a\{b\}_n$$

pour tout $a \in L$ et $b \in L^0$, et montrons que ceci induit une structure de groupe sur $1+L^0$. Ce groupe est appelé le groupe de jauge associé à L. Comme sur un corps de caractéristique nulle, nous montrons que ce groupe agit sur les éléments de Maurer-Cartan.

Théorème A. *Soit $\mathbb{K}$ un anneau.*

- *Pour toute dg algèbre pré-Lie à puissances divisées complète, le produit circulaire confère une structure de groupe à $1+L^0$ appelé* groupe de jauge.
- *Soit d la différentielle de L. Alors le groupe de jauge agit sur les éléments de Maurer-Cartan par*

$$(1+\mu)\cdot\alpha = (\alpha + \mu\{\alpha\}_1 - d(\mu)) \circledcirc (1+\mu)^{\circledcirc -1}$$

pour tout $\mu \in L^0, \alpha \in \mathcal{MC}(L)$.

Nous montrons également que cette nouvelle théorie de la déformation satisfait un analogue au théorème de Goldman-Millson ([GM88, §2.4]). Soit $\mathrm{Deligne}(L,A)$ le groupoïde de Deligne de la dg algèbre pré-Lie à puissances divisées $L\otimes\mathfrak{m}_A$, où L est une dg algèbre pré-Lie à puissances divisées et $\mathfrak{m}_A$ l'idéal maximal d'une algèbre locale artiniène A au-dessus du corps de fraction $\mathbf{K}$ de $\mathbb{K}$. Nous avons le résultat suivant.

Théorème B. *Soit $\mathbb{K}$ un anneau noethérien intègre et $\boldsymbol{K}$ son corps de fractions. Soit L et $\overline{L}$ deux $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algèbres positivement graduées. Soit $\varphi : L \longrightarrow \overline{L}$ un morphisme de $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algèbre tel que $H^0(\varphi)$ et $H^1(\varphi)$ sont des isomorphismes et $H^2(\varphi)$ un monomorphisme. Alors pour toute $\boldsymbol{K}$-algèbre locale artiniène A, le foncteur induit $\varphi_* : \mathrm{Deligne}(L,A) \longrightarrow \mathrm{Deligne}(\overline{L},A)$ est une équivalence de groupoïdes.*

La principale motivation de l'approche développée dans ce chapitre est que le dg $\mathbb{K}$-module $\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}})$ admet une structure de dg algèbre pré-Lie à puissances divisées. Nous obtenons alors le résultat suivant.

Théorème C. *Soit $\mathbb{K}$ un corps. Soit $\mathcal{C}$ est une coopérade Σ_*-cofibrante coaugmentée et $\mathcal{P}$ une opérade augmentée. Nous avons alors une bijection*

$$\pi_0\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}),\mathcal{P}) = \pi_0\mathrm{Deligne}(\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}})),$$

où $\pi_0\mathrm{Deligne}(\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}}))$ désigne l'ensemble des classes d'isomorphismes du groupoïde de Deligne.

Résultats du chapitre 2

Le calcul des groupes d'homotopies de $\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}),\mathcal{P})$ effectué dans [Yal16] utilise un repère simplicial explicite $\mathcal{P}^{\Delta^\bullet} = \mathcal{P}\otimes\Omega^*(\Delta^\bullet)$. Cependant, ce repère simplicial ne satisfait plus de bonnes propriétés homotopiques si la caractéristique de $\mathbb{K}$ est différente de 0. Ceci provient, entre autre, du fait que la cohomologie de $\Omega^*(\Delta^n)$ n'est pas réduite à 0 pour tout $n \geq 0$.

Le but de ce chapitre est de décrire $\mathrm{Map}^h_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P})$ par des équations de Maurer-Cartan, et de généraliser les résultats obtenus pour le calcul du π_0 dans le chapitre 1. Nous traitons principalement le cas des opérades non-symétriques, et expliquons ensuite comment généraliser au cas symétrique.

Puisque nous n'avons plus de repère simplicial explicite, nous utilisons un repère cosimplicial $B^c(\mathcal{C}) \otimes \Delta^\bullet$ associé à la construction cobar d'une coopérade non-symétrique coaugmentée $\mathcal{C}$ afin de calculer $\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}), \mathcal{P})$. Pour tout $n \geq 0$, nous construisons une dérivation tordue explicite ∂^n sur l'opérade libre $\mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1} N_*(\Delta^n))$ telle que

$$B^c(\mathcal{C}) \otimes \Delta^\bullet := (\mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1} N_*(\Delta^\bullet)), \partial^\bullet)$$

est un repère cosimplicial associé à $B^c(\mathcal{C})$. La raison de ce choix pour ce repère cosimplicial est que l'ensemble cosimplicial $M \otimes \Delta^\bullet := M \otimes N_*(\Delta^\bullet)$ définit un repère cosimplicial dans la catégorie des suites de dg $\mathbb{K}$-modules, puisque $N_*(\Delta^\bullet)$ est contractile.

Ainsi, les éléments de $\mathrm{Mor}_{\mathcal{O}p}(B^c(\mathcal{C}) \otimes \Delta^n, \mathcal{P})$ sont en correspondance bijective avec les éléments de $\Sigma\mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}}) \otimes N^*(\Delta^n)$ qui satisfont certaines équations. Notre but est d'interpréter ces équations comme des équations de Maurer-Cartan. Nos principales idées sont les suivantes. Nous utilisons des structures de $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algèbre, où $\mathcal{P}re\mathcal{L}ie_\infty$ est une opérade qui contrôle les algèbres pré-Lie à homotopie près. Nous verrons que nous pouvons étendre les équations de Maurer-Cartan aux $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algèbres. Le point clé est donc que si A est une algèbre brace et N une algèbre sous l'opérade de Barratt-Eccles $\mathcal{E}$, alors $A \otimes N$ est une $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algèbre. En utilisant ce résultat avec $A = \mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})$, qui est une algèbre brace d'après [LV12, Proposition 6.4.2] et [GV95, Proposition 1], et $N = N^*(\Delta^n)$, on obtient précisément les équations recherchées. Voici la mise en œuvre détaillée de ces idées.

Les $\mathcal{P}re\mathcal{L}ie_\infty$-algèbres, aussi appelées algèbres pré-Lie à homotopie près, on été étudiées dans [CL01]. Les auteurs montrent que se donner une structure d'algèbre pré-Lie à homotopie est équivalent à se donner des opérations brace qui satisfont certaines équations. Nous désignons ces opérations par $-\{\!\{-, \ldots, -\}\!\}$ dans cette thèse, et nous adoptons la convention que ces opérations sont définies sur ΣL. En suivant le même schéma de raisonnement que dans [Ces18] pour l'étude de la monade $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$, nous montrons que se donner une structure de $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algèbre sur L est équivalent à se donner des opérations braces à poids $-\{\!\{-, \ldots, -\}\!\}_{r_1, \ldots, r_n}$, définies pour toute collection d'entiers $r_1, \ldots, r_n \geq 0$ sur la suspension ΣL, qui sont similaires aux opérations

$$x\{\!\{y_1, \ldots, y_n\}\!\}_{r_1, \ldots, r_n} = \frac{1}{\prod_i r_i!} x\{\!\{\underbrace{y_1, \ldots, y_1}_{r_1}, \ldots, \underbrace{y_n, \ldots, y_n}_{r_n}\}\!\}.$$

Nous donnons également une autre caractérisation de cette structure, qui donnera un sens naturel à la notion de ∞-morphisme. Soit

$$\Gamma\mathrm{Perm}^c(V) = \bigoplus_{n \geq 0} V \otimes (V^{\otimes n})^{\Sigma_n}.$$

Nous montrons que, pour tout $\mathbb{K}$-module gradué V, cet espace est muni d'un coproduit $\Delta_{\Gamma\mathrm{Perm}}$ qui généralise le coproduit défini dans [CL01, Lemme 2.3] sur $\mathrm{Perm}^c(V) = V \otimes \mathcal{S}(V)$. Nous définissons alors la catégorie $\Gamma\Lambda\mathcal{PL}_\infty$ formée des couples (V, Q) tels que V est un $\mathbb{K}$-module gradué et Q une codérivation sur $\Gamma\mathrm{Perm}^c(V)$ de degré -1 telle que $Q^2 = 0$. Un morphisme dans $\Gamma\Lambda\mathcal{PL}_\infty$, aussi appelé ∞-morphisme, est un morphisme de cogèbres qui préserve les codérivations. On prouve que L est muni d'une structure de $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algèbre si et seulement si $\Sigma L \in \Gamma\Lambda\mathcal{PL}_\infty$.

On peut également définir une notion d'élément de Maurer-Cartan dans une $\Gamma\Lambda\mathcal{PL}_\infty$-algèbre, sous certaines hypothèses de convergence que nous détaillons dans ce chapitre. Cette hypothèse de convergence permet de donner un sens à l'équation dite de Maurer-Cartan

$$d(x) + \sum_{n\geq 1} x\{\!\{x\}\!\}_n = 0,$$

où x est un élément de degré 0 d'une $\Gamma\Lambda\mathcal{PL}_\infty$-algèbre V. La catégorie des $\Gamma\Lambda\mathcal{PL}_\infty$-algèbres satisfaisant cette hypothèse de convergence est notée $\widehat{\Gamma\Lambda\mathcal{PL}_\infty}$. On note $\mathcal{MC}(V)$ l'ensemble des éléments, dits de Maurer-Cartan, qui satisfont l'équation de Maurer-Cartan. On prouve que tout ∞-morphisme $\phi : V \longrightarrow V'$ entre deux $\widehat{\Gamma\Lambda\mathcal{PL}_\infty}$-algèbres induit une application $\mathcal{MC}(\phi) : \mathcal{MC}(V) \longrightarrow \mathcal{MC}(V')$ de sorte que $\mathcal{MC} : \widehat{\Gamma\Lambda\mathcal{PL}_\infty} \longrightarrow \mathrm{Set}$ soit un foncteur.

Théorème D. *Soit $\mathcal{B}race$ l'opérade des algèbres brace (voir [Cha02, Proposition 2]). Il existe un morphisme d'opérades $\mathcal{P}re\mathcal{L}ie_\infty \longrightarrow \mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$ qui rend le diagramme suivant commutatif :*

$$\begin{tikzcd} \mathcal{P}re\mathcal{L}ie_\infty \arrow[r] \arrow[d] & \mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E} \arrow[d] \\ \mathcal{P}re\mathcal{L}ie \arrow[r] & \mathcal{B}race \end{tikzcd}.$$

Puisque l'action de Σ_n sur $\mathcal{B}race(n) \otimes \mathcal{E}(n)$ est libre, Théorème D implique que toute $\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$-algèbre L est une $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algèbre via la composée

$$\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, L) \longrightarrow \Gamma(\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}, L) \xleftarrow{\simeq} \mathcal{S}(\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}, L) \longrightarrow L.$$

En utilisant que le complexe de cochaines normalisé $N^*(X)$ d'un ensemble simplicial X est muni d'une structure d'algèbre sous l'opérade de Barratt-Eccles (voir [BF04]) et Théorème D, on définit l'ensemble de Maurer-Cartan simplicial associé à une algèbre brace complète A par

$$\mathcal{MC}_\bullet(A) := \mathcal{MC}(A \otimes \Sigma N^*(\Delta^\bullet)).$$

En particulier, les sommets sont identifiés aux éléments de Maurer-Cartan de A, lorsqu'on utilise sa structure de $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algèbre sous-jacente (voir [Ver23, Theorem 2.15]). Nous calculons explicitement l'ensemble des composantes connexes et les groupes d'homotopie d'un tel ensemble simplicial. Rappelons pour cela que si A est une algèbre brace complète, alors tout élément de Maurer-Cartan $\tau \in \mathcal{MC}(A)$ induit une différentielle notée d_τ et définie par

$$d_\tau(x) = d(x) + \tau\langle x\rangle - (-1)^{|x|} x\langle\tau\rangle.$$

On notera A^τ le dg $\mathbb{K}$-module A munie de la différentielle d_τ. Nous avons le théorème suivant.

Théorème E. *Pour toute algèbre brace complète A, l'ensemble simplicial $\mathcal{MC}_\bullet(A)$ est un complexe de Kan. De plus, pour tout $\tau \in \mathcal{MC}(A)$, nous avons les bijection et isomorphismes suivants.*

— $\pi_0(\mathcal{MC}_\bullet(A)) \simeq \pi_0\mathrm{Deligne}(A)$;

— $\pi_1(\mathcal{MC}_\bullet(A), \tau) \simeq \{h \in A_0 \mid d(h) = \tau + h\langle\tau\rangle - \tau \circledcirc (1+h)\}/\sim_\tau$,
où $\sim_\tau$ est la relation d'équivalence suivante : $h \sim_\tau h'$ si et seulement si il existe $\psi \in A_1$ tel que

$$h - h' = d(\psi) + \psi\langle\tau\rangle + \sum_{p,q\geq 0} \tau\langle\underbrace{h,\ldots,h}_{p}, \psi, \underbrace{h',\ldots,h'}_{q}\rangle.$$

— $\pi_2(\mathcal{MC}_\bullet(A), \tau) \simeq (H_1(A^\tau), *_\tau)$
où $*_\tau$ est la structure de groupe suivante sur $H_1(A^\tau)$:

$$[\mu] *_\tau [\mu'] = [\mu + \mu' + \tau\langle\mu, \mu'\rangle].$$

— $\forall n \geq 3$, $\pi_{n+1}(\mathcal{MC}_\bullet(A), \tau) \simeq H_n(A^\tau)$.

Nous avons le résultat d'invariance homotopique suivant, qui étend le théorème de Goldman-Millson en dimension 0.

Théorème F. *Soit $\Theta : A \longrightarrow B$ un morphisme d'algèbres brace complètes tel que Θ est une équivalence faible dans $\mathrm{dgMod}_{\mathbb{K}}$. Alors $\mathcal{MC}_\bullet(\Theta) : \mathcal{MC}_\bullet(A) \longrightarrow \mathcal{MC}_\bullet(B)$ est une équivalence faible d'ensembles simpliciaux.*

Nous utilisons cette nouvelle notion d'ensemble de Maurer-Cartan simplicial pour l'étude de l'homotopie d'espaces d'applications dans la catégorie des opérades non symétriques. Pour tout $n \geq 0$, et pour toute coopérade coaugmentée non symétrique $\mathcal{C}$ telle que $\mathcal{C}(0) = 0$, nous construisons une dérivation tordue ∂^n sur $\mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1}N_*(\Delta^n))$ telle que l'opérade cosimpliciale

$$B^c(\mathcal{C}) \otimes \Delta^\bullet := (\mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1}N_*(\Delta^\bullet)), \partial^\bullet)$$

soit un repère cosimplicial associé à $B^c(\mathcal{C})$. Ceci mène au théorème suivant.

Théorème G. *Soit $\mathcal{C}$ une coopérade coaugmentée non symétrique et $\mathcal{P}$ une opérade augmentée non symétrique telles que $\mathcal{C}(0) = \mathcal{P}(0) = 0$ et $\mathcal{C}(1) = \mathcal{P}(1) = \mathbb{K}$. Nous avons l'isomorphisme suivant d'ensembles simpliciaux :*

$$\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}), \mathcal{P}) \simeq \mathcal{MC}_\bullet(\mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})).$$

Le calcul des composantes connexes et des groupes d'homotopies de $\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}), \mathcal{P})$ peuvent alors être effectués en utilisant Théorème E.

Dans le cas symétrique, la dérivation ∂^n précédemment construite sur $\mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1}N_*(\Delta^n))$ ne préserve pas l'action des groupes symétriques pour tout $n \geq 2$. Nous considérons un remplacement Σ_*-cofibrant de $B^c(\mathcal{C})$ donné par l'équivalence faible $B^c(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}}) \xrightarrow{\sim}$

$B^c(\mathcal{C})$, où $\mathbf{Sur}_{\mathbb{K}}$ désigne la coopérade des surjections définie dans [BCN23, Theorem A.1]. En utilisant que l'action de Σ_n sur $\overline{\mathcal{C}}(n)\otimes\overline{\mathbf{Sur}}_{\mathbb{K}}(n)$ est libre pour tout $n \geq 1$, nous construisons une dérivation tordue ∂^n sur $\mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1}N_*(\Delta^n))$ pour tout $n \geq 0$ telle que

$$B^c(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}}) \otimes \Delta^\bullet := (\mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1}N_*(\Delta^\bullet)), \partial^\bullet)$$

est un repère cosimplicial associé à $B^c(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}})$. Nous en déduisons le théorème suivant.

Théorème H. *Soit $\mathcal{C}$ une coopérade symétrique coaugmentée et $\mathcal{P}$ une opérade symétrique augmentée telles que $\mathcal{C}(0) = \mathcal{P}(0) = 0$ et $\mathcal{C}(1) = \mathcal{P}(1) = \mathbb{K}$. Alors $\Sigma\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes N_*(\Delta^\bullet), \overline{\mathcal{P}})$ est munie d'une structure de $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algèbre telle que nous avons un isomorphisme d'ensembles simpliciaux*

$$\mathrm{Map}^h_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P}) \simeq \mathcal{MC}(\Sigma\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes N_*(\Delta^\bullet), \mathcal{P})),$$

où $\mathrm{Map}^h_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P}) := \mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}}), \mathcal{P})$.

Organisation de la thèse

Cette thèse est composée de deux chapitres rédigés en anglais :

- Le chapitre 1, qui est une version sans modification de l'article *Pre-Lie algebras with divided powers and the Deligne groupoid in positive characteristic* [Ver23] à paraître dans la revue *Algebraic and Geometric Topology*,
- Le chapitre 2, qui fait suite au chapitre 1 et qui servira de base pour un mémoire ultérieur,

et d'une annexe qui fournit des rappels détaillés sur la théorie des opérades.

Conventions générales

Dans cette thèse, nous utilisons des $\mathbb{K}$-modules différentielles graduées (ou dg $\mathbb{K}$-modules) sur un anneau $\mathbb{K}$.

Dans le chapitre 1, nous adoptons des conventions cohomologiques, car nous souhaitons généraliser des résultats exprimés dans ce contexte dans la littérature. Dans ce contexte, un dg $\mathbb{K}$-module est un $\mathbb{K}$-module V muni d'une décomposition en somme directe de sous-modules $V \simeq \bigoplus_{n\in\mathbb{Z}} V^n$ et équipé d'une différentielle $d : V \longrightarrow V$ telle que $d(V^n) \subset V^{n+1}$ pour tout $n \in \mathbb{Z}$ et $d^2 = 0$.

Dans le chapitre 2, nous adoptons des conventions homologiques afin de donner des applications en théorie de l'homotopie. Dans ce contexte, un dg $\mathbb{K}$-module est un $\mathbb{K}$-module V muni d'une décomposition en somme directe de sous-modules $V \simeq \bigoplus_{n\in\mathbb{Z}} V_n$ et équipé d'une différentielle $d : V \longrightarrow V$ telle que $d(V_n) \subset V_{n-1}$ pour tout $n \in \mathbb{Z}$ et $d^2 = 0$.

Rappelons que nous avons une équivalence entre ces deux notions. Plus précisément, soit $V \simeq \bigoplus_{n\in\mathbb{Z}} V_n$ un dg $\mathbb{K}$-module de différentielle d en convention homologique. Si nous posons $V^n := V_{-n}$ pour tout $n \in \mathbb{Z}$, alors $V \simeq \bigoplus_{n\in\mathbb{Z}} V^n$ et $d(V^n) \subset V^{n+1}$ pour tout $n \in \mathbb{Z}$.

Un morphisme de dg $\mathbb{K}$-modules est un morphisme de $\mathbb{K}$-modules $f : V \longrightarrow W$ qui préserve la graduation, de sorte que $f(V_n) \subset W_n$ pour tout $n \in \mathbb{Z}$ (en convention homologique), et les différentielles de sorte que $fd = df$. Nous notons la catégorie des dg $\mathbb{K}$-modules $\mathrm{dgMod}_{\mathbb{K}}$ avec la convention de graduation (homologique ou cohomologique) fixée par le contexte.

Nous utilisons que la catégorie des dg $\mathbb{K}$-modules est équipée d'une structure de catégorie monoïdale symétrique. En convention homologique, si $V \simeq \bigoplus_{n\in\mathbb{Z}} V_n$ et $W \simeq \bigoplus_{n\in\mathbb{Z}} W_n$ sont des dg $\mathbb{K}$-modules, alors leur produit tensoriel $V \otimes W := V \otimes_{\mathbb{K}} W$ est aussi un dg $\mathbb{K}$-module avec $(V \otimes W)_n \simeq \bigoplus_{p+q=n} V_p \otimes W_q$ pour tout $n \in \mathbb{Z}$. La différentielle de $V \otimes W$ est défini par $d(v \otimes w) = d(v) \otimes w + (-1)^p v \otimes d(w)$ pour tout $v \in V_p$ et $w \in W$. L'opérateur de symétrie $\tau : V \otimes W \xrightarrow{\simeq} W \otimes V$ est définie par $\tau(x \otimes y) = (-1)^{pq} y \otimes x$ pour tout $x \in V_p$ et $y \in W_q$. Dans tout ce qui suit, nous adoptons la notation $\pm$ pour les signes que nous obtenons en appliquant cet opérateur. La règle générale (la règle de Koszul) est que toute permutation de facteurs $xy \longmapsto yx$ dans une expression multiplicative dans un dg $\mathbb{K}$-module produit un signe $\pm = (-1)^{pq}$ où p est le degré de x et q le degré de y.

Introduction (in English)

In this thesis, we study the homotopy of mapping spaces associated to operads defined over a field of positive characteristic.

Operads and algebras over operads

The notion of an operad has been introduced in order to define categories of algebras governed by operations. This notion has first been introduced for the study of loop spaces (see [May06]). Operads are now used in various branchs of mathematics.

For our purpose, we consider operads that are defined, generally, in a category of differential graded modules over a fixed ground ring $\mathbb{K}$ (the category of dg $\mathbb{K}$-modules for short). Fundamental examples of operads are defined in a category of modules. We can regard these operads as operads in dg $\mathbb{K}$-modules concentrated in degree zero. We work in the category of dg $\mathbb{K}$-modules in order to do homology theory and to address applications in homotopy theory.

In this setting, an operad is a sequence of dg $\mathbb{K}$-modules $(\mathcal{P}(n))_n$ such that, for every $n \geq 0$, the dg module $\mathcal{P}(n)$ is endowed with an action over the symmetric group on n letters Σ_n, and we have partial composition operations

$$\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \longrightarrow \mathcal{P}(n+m-1)$$

for every $1 \leq i \leq n$, which satisfy associativity, unit and equivariance axioms. For every $p \in \mathcal{P}(n)$ and $q \in \mathcal{P}(m)$, the operation $p \circ_i q$ represents the insertion of the operation q in the i-th input of p. An operad morphism $\mathcal{P} \longrightarrow \mathcal{Q}$ is a sequence of morphisms $\mathcal{P}(n) \longrightarrow \mathcal{Q}(n)$ which preserve the operadic compositions of $\mathcal{P}$ and $\mathcal{Q}$ and the action of the symmetric groups.

The first example of an operad is the endomorphism operad End_V associated to a dg $\mathbb{K}$-module V. This operad is defined by $\mathrm{End}_V(n) = \mathrm{Hom}(V^{\otimes n}, V)$ for every $n \geq 0$, where the i-th composition of $f \in \mathrm{End}_V(p)$ and $g \in \mathrm{End}_V(q)$ is defined by

$$(f \circ_i g)(v_1 \otimes \cdots \otimes v_{p+q-1}) = \pm f(v_1 \otimes \cdots \otimes v_{i-1} \otimes g(v_i \otimes \cdots \otimes v_{i+q-1}) \otimes v_{i+q} \otimes \cdots \otimes v_{p+q-1})$$

for every $v_1, \ldots, v_{p+q-1} \in V$.

For $\mathcal{P}$ an operad, a $\mathcal{P}$-algebra is a dg $\mathbb{K}$-module A endowed with morphisms

$$\mathcal{P}(n) \otimes A^{\otimes n} \longrightarrow A$$

compatible with the operadic composition of $\mathcal{P}$. The elements of $\mathcal{P}$ are then viewed as morphisms $A^{\otimes n} \longrightarrow A$. Thus, giving a $\mathcal{P}$-algebra structure on A is equivalent to giving an operad morphism $\mathcal{P} \longrightarrow \mathrm{End}_A$.

This formalism allows us to recover classical algebraic structures. For instance, there is an operad $\mathcal{C}om$, which governs associative and commutative algebras, an operad $\mathcal{A}s$, which governs associative algebras, an operad $\mathcal{L}ie$, which governs Lie algebras...

In this thesis, we often need to give a sense to infinite sums that can occur in some Maurer-Cartan equations. To achieve this, we use a notion of a complete $\mathcal{P}$-algebra, which we form in a category of complete filtered dg $\mathbb{K}$-modules. A filtration on a dg $\mathbb{K}$-module V is a sequence of dg $\mathbb{K}$-modules $(F_nV)_{n\geq 1}$ such that

$$\cdots \subset F_nV \subset F_{n-1}V \subset \cdots \subset F_1V = V.$$

The completion of V for this filtration is defined by $\widehat{V} := \lim_{n\geq 1} V/F_nV$. The dg $\mathbb{K}$-module V is said to be complete for its underlying filtration if $V \simeq \widehat{V}$. In particular, for every dg $\mathbb{K}$-module V endowed with a filtration, the completion $\widehat{V}$ is complete. If W is an other filtered dg $\mathbb{K}$-module, the tensor product $V \otimes W$ over $\mathbb{K}$ is also endowed with a filtration defined by

$$F_n(V \otimes W) = \bigoplus_{p+q=n} F_pV \otimes F_qW.$$

In general, even if V and W are complete for their underlying filtrations, the tensor product $V \otimes W$ is not complete. We therefore define the complete tensor product by $V \widehat{\otimes} W := \widehat{V \otimes W}$, when considering the above filtration on $V \otimes W$. A complete $\mathcal{P}$-algebra is then a $\mathcal{P}$-algebra endowed with a filtration for which it is complete and such that, for every $p \in \mathcal{P}(n)$, the morphism $V^{\otimes n} \longrightarrow V$ induced by p preserves the filtrations of $V^{\otimes n}$ and V.

Simplicial mapping spaces

The notion of a simplicial mapping space is defined in a very general framework. The idea is to formalize properties that reflect a simplicial model of the mapping spaces of topology. In what follows, we just use the phrase "mapping space" for "simplicial mapping space" since we only use this simplicial version of the notion of a mapping space.

The category of topological spaces $\mathcal{T}op$ comes equipped with a functor $\mathrm{Map}_{\mathcal{T}op}(-,-) : \mathcal{T}op^{op} \times \mathcal{T}op \longrightarrow \mathrm{sSet}$ which makes $\mathcal{T}op$ a category enriched over the category of simplicial sets. This functor can be used in order to encode notions of higher homotopy in the category of $\mathcal{T}op$. For every topological spaces X, Y, the connected components of $\mathrm{Map}_{\mathcal{T}op}(X, Y)$ are in bijection with the homotopy classes of morphisms $X \longrightarrow Y$, while the homotopy groups encode higher homotopy relations. This approach allows us to use tools from algebraic topology in order to study morphisms up to homotopy in $\mathcal{T}op$.

The simplicial set $\mathrm{Map}_{\mathcal{T}op}(X,Y)$ is defined as follows. For every $X \in \mathcal{T}op$, we define two functors $X \otimes - : \mathrm{sSet} \longrightarrow \mathcal{T}op$ and $X^{-} : \mathrm{sSet}^{op} \longrightarrow \mathcal{T}op$ by

$$X \otimes K := X \times |K| \quad ; \quad X^K := \mathrm{Mor}_{\mathcal{T}op}(|K|, X),$$

for every $K \in \mathrm{sSet}$, where $|K|$ denotes the geometric realization of the simplicial set K. For every $X, Y \in \mathcal{T}op$ and $K \in \mathrm{sSet}$, we have the isomorphism

$$\mathrm{Mor}_{\mathcal{T}op}(X \otimes K, Y) \simeq \mathrm{Mor}_{\mathcal{T}op}(X, Y^K).$$

We thus set $\mathrm{Map}_{\mathcal{T}op}(X,Y) := \mathrm{Mor}_{\mathcal{T}op}(X \otimes \Delta^\bullet, Y)$, where, for every $n \geq 0$, we denote by Δ^n the fundamental n-simplex.

In a general model category C, it is possible to partially generalize these results. For every $X, Y \in C$, there exists a cosimplicial object $X \otimes \Delta^\bullet$ called cosimplicial frame associated to X, and a simplicial object $Y^{\Delta^\bullet}$ called simplicial frame associated to Y such that the two simplicial sets $\mathrm{Mor}_C(X \otimes \Delta^\bullet, Y)$ and $\mathrm{Mor}_C(X, Y^{\Delta^\bullet})$ are Kan complexes related by a zig-zag of weak equivalences. We can define a simplicial set $\mathrm{Map}_C(X,Y)$ by $\mathrm{Map}_C(X,Y) = \mathrm{Mor}_C(X \otimes \Delta^\bullet, Y)$, or equivalently by $\mathrm{Map}_C(X,Y) = \mathrm{Mor}_C(X, Y^{\Delta^\bullet})$, so that we still have a bijection between its connected components and the homotopy classes of morphisms $X \longrightarrow Y$.

We are interested in the case where C is either the category of non-symmetric operads, or the category of connected symmetric operads. In these cases, the study of the simplicial set $\mathrm{Map}_{\mathcal{O}p}(\mathcal{Q}, \mathcal{P})$ allows us to have a better understanding of operad morphisms $\mathcal{Q} \longrightarrow \mathcal{P}$ up to homotopy. The motivation for the study of these objects in the categories of operads is that if we set $\mathcal{P} = \mathrm{End}_V$, then the space $\mathrm{Map}_{\mathcal{O}p}(\mathcal{Q}, \mathrm{End}_V)$ determines the homotopy of a moduli space of $\mathcal{Q}$-algebra structures on V.

State-of-the-art in characteristic 0

Let $\mathcal{Q} = B^c(\mathcal{C})$ be the cobar construction of a coaugmented cooperad $\mathcal{C}$ such that $\mathcal{C}(0) = 0$ and let $\mathcal{P}$ be an augmented operad. The study of the homotopy of $\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P})$ is already known when $\mathbb{K}$ is of characteristic 0. The computation of the homotopy groups has been addressed in [Yal16], in the properadic framework, by using an explicit simplicial frame associated to $\mathcal{P}$. This explicit simplicial frame is defined by

$$\mathcal{P}^{\Delta^\bullet} := \mathcal{P} \otimes \Omega^*(\Delta^\bullet)$$

where, for every $n \geq 0$, the dg $\mathbb{K}$-module $\Omega^*(\Delta^n)$ denotes the Sullivan algebra of polynomial de Rham forms on Δ^n (see for instance [BG76, §2.1]). We then obtain

$$\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P}) = \mathrm{Mor}_{\Sigma\mathcal{O}p}(\mathcal{Q}, \mathcal{P}^{\Delta^\bullet}).$$

The n-simplices of $\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P})$ are then identified with elements of the tensor product $\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}}) \widehat{\otimes} \Omega^*(\Delta^n)$ which satisfy some equations, where $\overline{\mathcal{C}}$ is the coaugmentation coideal of $\mathcal{C}$ and $\overline{\mathcal{P}}$ the augmentation ideal of $\mathcal{P}$.

These equations, called Maurer-Cartan equations, can be described by using a Lie algebra structure on $\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})\widehat{\otimes}\Omega^*(\Delta^n)$. This Lie algebra structure can be deduced from a pre-Lie algebra structure on $\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})$. Here are the details of this construction.

Recall that if L is a complete Lie algebra, then a Maurer-Cartan element is an element $\tau \in L_{-1}$ such that

$$d(\tau) + \frac{1}{2}[\tau, \tau] = 0.$$

We denote by $\mathcal{MC}(L)$ the set of Maurer-Cartan elements. Every Maurer-Cartan element $\tau \in \mathcal{MC}(L)$ induces a differential d_τ on L defined by

$$d_\tau(x) = d(x) + [\tau, x]$$

for every $x \in L$. We denote by L^τ the dg $\mathbb{K}$-module L endowed with the differential d_τ. By using the commutative algebra structure on $\Omega^*(\Delta^n)$ for every $n \geq 0$, we can endow the dg $\mathbb{K}$-module $L\widehat{\otimes}\Omega^*(\Delta^n)$ with the structure of a complete Lie algebra. We then define the simplicial Maurer-Cartan set associated to L by

$$\mathcal{MC}_\bullet(L) = \mathcal{MC}(L\widehat{\otimes}\Omega^*(\Delta^\bullet)).$$

Pre-Lie algebras are examples of Lie algebras. A pre-Lie algebra is a dg $\mathbb{K}$-module L endowed with an operation $\star : L \otimes L \longrightarrow L$ such that

$$(x \star y) \star z - x \star (y \star z) = (-1)^{|y||z|}((x \star z) \star y - x \star (z \star y))$$

for every $x, y, z \in L$. The Lie bracket induced by the pre-Lie structure is given by

$$[x, y] = x \star y - (-1)^{|x||y|} y \star x.$$

The pre-Lie algebra structure of $\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})$ is given by the composite

$$f \star g : \overline{\mathcal{C}} \xrightarrow{\Delta_{(1)}} \overline{\mathcal{C}} \circ_{(1)} \overline{\mathcal{C}} \xrightarrow{\overline{f} \circ_{(1)} \overline{g}} \overline{\mathcal{P}} \circ_{(1)} \overline{\mathcal{P}} \xrightarrow{\gamma_{(1)}} \overline{\mathcal{P}}$$

for every $f, g \in \mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})$, where $\Delta_{(1)}, \circ_{(1)}$ and $\gamma_{(1)}$ denote the infinitesimal components of operadic and cooperadic composition structures (see [LV12, §6.4.4]).

The result is that we have a bijection between the operad morphisms $B^c(\mathcal{C}) \longrightarrow \mathcal{P}^{\Delta^n}$ and the Maurer-Cartan elements of $\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})\widehat{\otimes}\Omega^*(\Delta^n)$ for every $n \geq 0$, so that

$$\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P}) = \mathcal{MC}_\bullet(\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})).$$

The computation of the homotopy groups of $\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P})$ can then be achieved by the general computation of the homotopy groups of the simplicial Maurer-Cartan set $\mathcal{MC}_\bullet(L)$ associated to a Lie algebra L (see [Ber15, Theorem 1.1]). Explicitly, for every $k \geq 0$ and $\tau \in \mathcal{MC}(L)$,

$$\pi_{k+1}(\mathcal{MC}_\bullet(L), \tau) \simeq H_k(L^\tau),$$

where $H_0(L^\tau)$ is endowed with the group structure given by the Baker-Campbell-Hausdorff formula.

The computation of the connected components of $\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P})$ can be achieved as follows. We first use the formal identification

$$\pi_0\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P}) = \mathrm{Mor}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P})/\sim$$

where $\sim$ denotes the homotopy equivalence relation in the model category $\Sigma\mathcal{O}p$ (see [Fre17b, Theorem 3.2.14]). The computation of the right hand-side term can be made by using the pre-Lie deformation theory developed in [DSV16] which generalizes the Lie deformation theory. Explicitly, a Maurer-Cartan element τ in a pre-Lie algebra L is an element $\tau \in L_{-1}$ such that

$$d(\tau) + \tau \star \tau = 0.$$

The gauge group $(L_0, BCH, 0)$ can also be described in terms of pre-Lie operations. Consider the subset $1 + L^0 \subset \mathbb{K} \oplus L$. Under convergence hypothesis, we define the circular product $\circledcirc : L \otimes (1 + L_0) \longrightarrow L$ by

$$x \circledcirc (1 + y) = \sum_{n\geq 0} \frac{1}{n!} x\{\underbrace{y, \dots, y}_{n}\},$$

for every $x \in L$ and $y \in L_0$, where we denote by $-\{-, \dots, -\}$ the symmetric brace operations in the pre-Lie algebra L (see [OG08] or [LM05]). We can restrict this circular product to an operation on $1 + L_0$ defined by

$$(1 + x) \circledcirc (1 + y) = 1 + y + \sum_{n\geq 0} \frac{1}{n!} x\{\underbrace{y, \dots, y}_{n}\}$$

for every $x, y \in L_0$. Then the triple $(1 + L_0, \circledcirc, 1)$ is a group which is isomorphic to the gauge group (see [DSV16, Theorem 2]). The group $(1 + L_0, \circledcirc, 1)$ also acts on $\mathcal{MC}(L)$ via the formula

$$(1 + \mu) \cdot \tau = (\tau + \mu \star \tau - d(\mu)) \circledcirc (1 + \mu)^{-1}$$

for every $\mu \in L_0$ and $\tau \in \mathcal{MC}(L)$. We then define the Deligne groupoid, denoted by $\mathrm{Deligne}(L)$, as the category with $\mathcal{MC}(L)$ as set objects, and the elements $1 + L_0$ as morphisms.

Recall that $B^c(\mathcal{C})$, for $\mathcal{C}$ is a coaugmented cooperad, is the dg-operad such that $B^c(\mathcal{C}) = (\mathcal{F}(\Sigma^{-1}\overline{\mathcal{C}}), \partial)$, where we take the free operad functor $\mathcal{F}$ on the desuspension of the coaugmentation coideal of our cooperad $\mathcal{C}$, and ∂ a derivation, determined by the partial composition coproducts of $\mathcal{C}$, which is added to the internal differential of $\mathcal{C}$ to produce the differential of $B^c(\mathcal{C})$. The homotopy relation for morphisms $B^c(\mathcal{C}) \longrightarrow \mathcal{P}$ can be computed by using an explicit cylinder object associated to $B^c(\mathcal{C})$ given in [Fre17b, Theorem 3.2.14]. We have an explicit twisting derivation ∂ on $\mathcal{F}(\Sigma^{-1}\overline{\mathcal{C}} \otimes N_*(\Delta^1))$ such that

$$B^c(\mathcal{C}) \otimes \Delta^1 := (\mathcal{F}(\Sigma^{-1}\overline{\mathcal{C}} \otimes N_*(\Delta^1)), \partial)$$

is a cylinder object associated to $B^c(\mathcal{C})$, where $N_*(\Delta^1)$ is the normalized cochain complex associated to the simplicial set Δ^1. By writing the commutation condition with the differential ∂, two morphisms $f, g : B^c(\mathcal{C}) \longrightarrow \mathcal{P}$ are related by a homotopy if and only if their corresponding elements in $\mathcal{MC}(\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}}))$ are in the same orbit under the action of the gauge group of $\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}})$ (see [DSV16, Corollary 2]). We thus obtain a bijection

$$\pi_0\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}),\mathcal{P}) \simeq \pi_0\mathrm{Deligne}(\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}}))$$

where the right hand-side term denotes the set of isomorphism classes of the groupoid $\mathrm{Deligne}(\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}}))$.

Issues in positive characteristic and idea of the thesis

The purpose of this thesis is to generalize this approach in positive characteristic. Several issues occur when considering a ground ring $\mathbb{K}$ with positive characteristic.

First, all the above computations (especially for the computation of the connected components) use rational coefficients. Our idea is to use (some generalized) divided power operations in order to handle the non-trivial denominators that occur in the formulas. For instance, in Chapter 1, we use divided power structures on pre-Lie algebras in order to address the computation of the connected components of $\mathrm{Map}_{\Sigma\mathcal{O}p}(B^c(\mathcal{C}),\mathcal{P})$. In Chapter 2, we use divided powers structures on pre-Lie algebras up to homotopy in order to address the computation of the homotopy type of $\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}),\mathcal{P})$.

Second, the simplicial frame $\mathcal{P}^{\Delta^\bullet} = \mathcal{P}\otimes\Omega^*(\Delta^\bullet)$, which we use in order to compute the homotopy groups, is no longer a simplicial frame when $\mathbb{K}$ is a field with positive characteristic. The reason is, among others, that the cohomology of $\Omega^*(\Delta^\bullet)$ is non-zero when the ground field has a positive characteristic. In Chapter 2, in the framework of non-symmetric operads, we instead define a cosimplicial frame $B^c(\mathcal{C})\otimes\Delta^\bullet$ associated to the cobar construction of a non-symmetric coaugmented cooperad $\mathcal{C}$. We use this explicit cosimplicial frame in order to compute $\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}),\mathcal{P})$.

Results of Chapter 1

The deformation theory developed in [DSV16] does not allow us to identify $\pi_0\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}),\mathcal{P})$ with the set of isomorphism classes of a Deligne groupoid if $char(\mathbb{K}) > 0$, as the formulas which define the gauge group associated to a pre-Lie algebra L, and its action of the Maurer-Cartan elements, are given by formulas with rational coefficients.

In this first chapter, we develop a deformation theory which generalizes the deformation theory controlled by pre-Lie algebras described in [DSV16] over a field with positive characteristic, and apply this to the computation of $\pi_0\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}),\mathcal{P})$. Our idea is to use pre-Lie algebras with divided powers.

The notion of $\mathcal{P}$-algebras with divided powers has been first introduced by Fresse in [Fre00], where $\mathcal{P}$ is an operad such that $\mathcal{P}(0) = 0$. Giving a $\mathcal{P}$-algebra structure on V is equivalent to giving an algebra structure over the monad $\mathcal{S}(\mathcal{P},-) : \mathrm{dgMod}_{\mathbb{K}} \longrightarrow$

$\mathrm{dgMod}_{\mathbb{K}}$ called the Schur functor and defined by

$$\mathcal{S}(\mathcal{P}, V) := \bigoplus_{n \geq 0} \mathcal{P}(n) \otimes_{\Sigma_n} V^{\otimes n}$$

where we consider the diagonal action of Σ_n on $\mathcal{P}(n) \otimes V^{\otimes n}$ given by its action on $\mathcal{P}(n)$ and its action on $V^{\otimes n}$ given by the permutation of the factors, and $\otimes_{\Sigma_n}$ denotes a module of coinvariants with respect to this diagonal action. If $\mathcal{P}(0) = 0$, then the analogous functor given by invariants

$$\Gamma(\mathcal{P}, V) := \bigoplus_{n \geq 1} \mathcal{P}(n) \otimes^{\Sigma_n} V^{\otimes n}$$

also admits the structure of a monad (see [Fre00, §1.1.18]). In this case, we have a monad morphism

$$Tr : \mathcal{S}(\mathcal{P}, -) \longrightarrow \Gamma(\mathcal{P}, -)$$

given by the usual trace map which starts from coinvariants to invariants, when considering the action of Σ_n on $\mathcal{P}(n) \otimes V^{\otimes n}$ for every $n \geq 1$. A $\mathcal{P}$-algebra with divided powers is an algebra over the monad $\Gamma(\mathcal{P}, -)$. In particular, every $\mathcal{P}$-algebra with divided powers is endowed with the structure of a $\mathcal{P}$-algebra induced by the trace map.

The notion of a pre-Lie algebra with divided powers (or $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra) has been studied by Cesaro in [Ces18]. He showed in particular that every pre-Lie algebra with divided powers comes equipped with weighted brace operations $-\{-, \ldots, -\}_{r_1, \ldots, r_n}$, for each collection of integers $r_1, \ldots, r_n \geq 0$, which satisfy similar identities as the quantities

$$x\{y_1, \ldots, y_n\}_{r_1, \ldots, r_n} = \frac{1}{\prod_i r_i!} x\{\underbrace{y_1, \ldots, y_1}_{r_1}, \ldots, \underbrace{y_n, \ldots, y_n}_{r_n}\}$$

in a pre-Lie algebra over a field of characteristic 0 (see [Ces18, Propositions 5.9-5.10] for a precise list of these identities).

Every differential graded pre-Lie algebra with divided powers L comes equipped with analogous weighted brace operations $-\{-, \ldots, -\}_{r_1, \ldots, r_n}$ which satisfy a graded version of the identities satisfied by weighted braces in the non graded framework. In this context, we have an analogue of the Maurer-Cartan equation:

$$d(x) + x\{x\}_1 = 0.$$

With suitable convergence hypothesis, we also get that the circular product $\circledcirc$ can be written as

$$a \circledcirc (1 + b) = \sum_{n \geq 0} a\{b\}_n$$

and gives rise to a group structure on $1 + L^0$. This group is called the *gauge group* of L. As over a field with characteristic 0, we also show that this group acts on the Maurer-Cartan set of L.

Theorem A. *Let $\mathbb{K}$ be a ring.*

(i) In any complete differential graded pre-Lie algebra with divided powers L, the circular product $\circledcirc$, defined as above, endows the set $1+L^0$ with a group structure.

(ii) If we denote by d the differential of L, then this group acts on the Maurer-Cartan set via the formula

$$(1+\mu)\cdot\alpha = (\alpha+\mu\{\alpha\}_1 - d(\mu)) \circledcirc (1+\mu)^{\circledcirc -1}.$$

We prove that this new deformation theory satisfies an analogue of the Goldman-Millson theorem given in [GM88, §2.4]. Let $\mathrm{Deligne}(L,A)$ be the Deligne groupoid of the complete dg pre-Lie algebra with divided powers $L\otimes\mathfrak{m}_A$, where L is a dg pre-Lie algebra with divided powers and $\mathfrak{m}_A$ the maximal ideal of a local artinian algebra A over the field of fraction $\mathbf{K}$ of $\mathbb{K}$. We precisely get the following result.

Theorem B. *Let $\mathbb{K}$ be a noetherian integral domain and $\boldsymbol{K}$ its field of fractions. Let L and $\overline{L}$ be two positively graded $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebras. Let $\varphi : L \longrightarrow \overline{L}$ be a morphism of $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebras such that $H^0(\varphi)$ and $H^1(\varphi)$ are isomorphisms and $H^2(\varphi)$ is a monomorphism. Then for every local artinian $\boldsymbol{K}$-algebra A, the induced functor $\varphi_* : \mathrm{Deligne}(L,A) \longrightarrow \mathrm{Deligne}(\overline{L},A)$ is an equivalence of groupoids.*

The main motivation for the approach developed in this paper is that the dg module $\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}})$ is endowed with the structure of a $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra. We then obtain the following.

Theorem C. *Let $\mathbb{K}$ be a field. Suppose that $\mathcal{C}$ is a Σ_*-cofibrant coaugmented dg cooperad and $\mathcal{P}$ an augmented dg operad. We then have an isomorphism:*

$$\pi_0\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}),\mathcal{P}) \simeq \pi_0\mathrm{Deligne}(\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}})),$$

where $\pi_0\mathrm{Deligne}(\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}}))$ denotes the set of isomorphism classes of the Deligne groupoid.

Results of Chapter 2

The computation of the homotopy groups of $\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}),\mathcal{P})$ done in [Yal16] uses an explicit simplicial frame $\mathcal{P}^{\Delta^\bullet} = \mathcal{P}\otimes\Omega^*(\Delta^\bullet)$. However, this explicit simplicial frame no longer satisfies the right homotopical properties if $char(\mathbb{K})\neq 0$. This is because the cohomology of $\Omega^*(\Delta^n)$ is non zero for every $n\geq 0$.

The goal of this chapter is to compute $\mathrm{Map}^h_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}),\mathcal{P})$ in terms of Maurer-Cartan elements and to generalize the results obtained for π_0 in Chapter 1. We mostly deal with the case of non-symmetric operads, and explain the generalization to the symmetric context afterwards.

As we no longer have an explicit simplicial frame, we use an explicit cosimplicial frame $B^c(\mathcal{C})\otimes\Delta^\bullet$ associated to the cobar construction of a coaugmented non-symmetric cooperad $\mathcal{C}$ in order to compute $\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}),\mathcal{P})$. For every $n\geq 0$, we will explicitly construct a twisting derivation ∂^n on the free operad $\mathcal{F}(\overline{\mathcal{C}}\otimes\Sigma^{-1}N_*(\Delta^n))$ such that

$$B^c(\mathcal{C})\otimes\Delta^\bullet := (\mathcal{F}(\overline{\mathcal{C}}\otimes\Sigma^{-1}N_*(\Delta^\bullet)),\partial^\bullet)$$

is a cosimplicial frame associated to $B^c(\mathcal{C})$. The reason for the choice of this cosimplicial frame is that the cosimplicial set $M \otimes \Delta^\bullet = M \otimes N_*(\Delta^\bullet)$ defines a cosimplicial frame in the category of sequences of dg $\mathbb{K}$-modules, as $N_*(\Delta^\bullet)$ is contractible.

Therefore, elements of $\mathrm{Mor}_{\mathcal{O}p}(B^c(\mathcal{C}) \otimes \Delta^n, \mathcal{P})$ are in bijective correspondence with elements of $\Sigma \mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}}) \otimes N^*(\Delta^n)$ which satisfy some equations. Our purpose is to interpret these equations as Maurer-Cartan equations. Our main ideas are the following. We deal with $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebra structures, where $\mathcal{P}re\mathcal{L}ie_\infty$ is the operad which governs pre-Lie algebras up to homotopy. We will see that we can extend the Maurer-Cartan equations to $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty,)$-algebras. Then, the key point is that if A is a brace algebra and if N is an algebra over the Barratt-Eccles operad $\mathcal{E}$, then $A \otimes N$ is a $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebra. Using this result with $A = \mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})$, which is a brace algebra by [LV12, Proposition 6.4.2] and [GV95, Proposition 1], and $N = N^*(\Delta^n)$ precisely give the desired equations. Here is the detailed implementation of these ideas.

The $\mathcal{P}re\mathcal{L}ie_\infty$-algebras, also called pre-Lie algebras up to homotopy, have been studied in [CL01]. The author characterized the data of a $\mathcal{P}re\mathcal{L}ie_\infty$-algebra structure on L as the data of brace operations which satisfy some identities. We denote these brace operations by $-\{\!\{-, \dots, -\}\!\}$ in this thesis, and we assume that these operations are defined on the suspension ΣL. As for the study of the monad $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$ in [Ces18], we prove that giving a $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebra structure on L is equivalent to giving weighted brace operations $-\{\!\{-, \dots, -\}\!\}_{r_1,\dots,r_n}$ on the suspension ΣL which are similar to the operations

$$x\{\!\{y_1, \dots, y_n\}\!\}_{r_1,\dots,r_n} = \frac{1}{\prod_i r_i!} x\{\!\{\underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n}\}\!\}.$$

We give another characterization of such objects that will emphasize a notion of ∞-morphism. For any graded $\mathbb{K}$-module V, we set

$$\Gamma\mathrm{Perm}^c(V) = \bigoplus_{n \geq 0} V \otimes (V^{\otimes n})^{\Sigma_n}.$$

We prove that $\Gamma\mathrm{Perm}^c(V)$ is endowed with a coproduct $\Delta_{\Gamma\mathrm{Perm}}$ which, in some sense, is compatible with the coproduct defined in [CL01, Lemma 2.3] on $\mathrm{Perm}^c(V) := V \otimes \mathcal{S}(V)$. We then define the category $\Gamma\Lambda\mathcal{PL}_\infty$ formed by pairs (V, Q) where V is a graded $\mathbb{K}$-module and Q a coderivation on $\Gamma\mathrm{Perm}^c(V)$ of degree -1 such that $Q^2 = 0$. A morphism in $\Gamma\Lambda\mathcal{PL}_\infty$, also called an ∞-morphism, is a morphism of coalgebras which preserve the coderivations. We prove that L is a $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebra if and only if $\Sigma L \in \Gamma\Lambda\mathcal{PL}_\infty$.

We can also define a notion of Maurer-Cartan elements in a $\Gamma\Lambda\mathcal{PL}_\infty$-algebra, under some convergence hypothesis which we detail in this chapter. This convergence hypothesis allows us to write the Maurer-Cartan equation

$$d(x) + \sum_{n \geq 1} x\{\!\{x\}\!\}_n = 0,$$

where x is a degree 0 element of some $V \in \Gamma\Lambda\mathcal{PL}_\infty$. We denote by $\widehat{\Gamma\Lambda\mathcal{PL}_\infty}$ the category of $\Gamma\Lambda\mathcal{PL}_\infty$-algebras which satisfy the required convergence hypothesis. We denote by

$\mathcal{MC}(V)$ the set of Maurer-Cartan elements. We prove that any ∞-morphism $\phi : V \rightsquigarrow W$ induces a set map $\mathcal{MC}(\phi) : \mathcal{MC}(V) \longrightarrow \mathcal{MC}(W)$ so that $\mathcal{MC} : \widehat{\Gamma\Lambda\mathcal{PL}_\infty} \longrightarrow \mathrm{Set}$ is a functor.

Theorem D. *Let $\mathcal{B}race$ be the operad which governs brace algebras (see [Cha02, Proposition 2]). There exists an operad morphism $\mathcal{P}re\mathcal{L}ie_\infty \longrightarrow \mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$ which fits in a commutative square*

$$\begin{tikzcd}
\mathcal{P}re\mathcal{L}ie_\infty \arrow[r] \arrow[d] & \mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E} \arrow[d] \\
\mathcal{P}re\mathcal{L}ie \arrow[r] & \mathcal{B}race
\end{tikzcd}.$$

As the action of Σ_n on $\mathcal{B}race(n)\otimes\mathcal{E}(n)$ is free for every $n \geq 0$, Theorem D implies that every $\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$-algebra L is a $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebra, via the composite

$$\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, L) \longrightarrow \Gamma(\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}, L) \xleftarrow{\simeq} \mathcal{S}(\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}, L) \longrightarrow L.$$

Using that the normalized cochain complex $N^*(X)$ of a simplicial set X admits the structure of an algebra over the Barratt-Eccles operad (see [BF04]) and Theorem D, we define the simplicial Maurer-Cartan set associated to a complete brace algebra A as

$$\mathcal{MC}_\bullet(A) = \mathcal{MC}(A \otimes \Sigma N^*(\Delta^\bullet)).$$

In particular, the vertices are identified with Maurer-Cartan elements in A, when using its underlying $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra structure (see [Ver23, Theorem 2.15]). We explicitly compute the connected components and the homotopy groups of $\mathcal{MC}_\bullet(A)$. Recall that if A is a complete brace algebra, then every Maurer-Cartan element $\tau \in \mathcal{MC}(A)$ induces a differential denoted by d_τ and defined by

$$d_\tau(x) = d(x) + \tau\langle x\rangle - (-1)^{|x|}x\langle\tau\rangle.$$

We denote by A^τ the dg $\mathbb{K}$-module A endowed with the differential d_τ. We have the following theorem.

Theorem E. *For every complete brace algebra A, the simplicial set $\mathcal{MC}_\bullet(A)$ is a Kan complex. Moreover, we have the bijection and isomorphisms for every $\tau \in \mathcal{MC}(A)$.*

- $\pi_0(\mathcal{MC}_\bullet(A)) \simeq \pi_0\mathrm{Deligne}(A)$, *where* $\mathrm{Deligne}(A)$ *denotes the Deligne groupoid associated to the* $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$*-algebra A (see [Ver23, Proposition-Definition 2.30]);*
- $\pi_1(\mathcal{MC}_\bullet(A), \tau) \simeq \{h \in A_0 \mid d(h) = \tau + h\langle\tau\rangle - \tau \circledcirc (1 + h)\}/\sim_\tau$, *where $\sim_\tau$ is the equivalence relation such that $h \sim_\tau h'$ if and only if there exists $\psi \in A_1$ such that*

$$h - h' = d(\psi) + \psi\langle\tau\rangle + \sum_{p,q\geq 0} \tau\langle\underbrace{h, \dots, h}_{p}, \psi, \underbrace{h', \dots, h'}_{q}\rangle.$$

- $\pi_2(\mathcal{MC}_\bullet(A), \tau) \simeq (H_1(A^\tau), *_\tau, 0)$, *where $*_\tau$ is the group structure on $H_1(A^\tau)$ such that*

$$[\mu] *_\tau [\mu'] = [\mu + \mu' + \tau\langle\mu, \mu'\rangle].$$

— $\pi_{n+1}(\mathcal{MC}_\bullet(A),\tau) \simeq H_n(A^\tau)$ *for every* $n \geq 3$.

We have the following homotopy invariance result, which extends the Goldman-Millson theorem in dimension 0.

Theorem F. *Let* $\Theta : A \longrightarrow B$ *be a morphism of complete brace algebras such that* Θ *is a weak equivalence in* $\mathrm{dgMod}_{\mathbb{K}}$*. Then* $\mathcal{MC}_\bullet(\Theta) : \mathcal{MC}_\bullet(A) \longrightarrow \mathcal{MC}_\bullet(B)$ *is a weak equivalence.*

We use this new notion of simplicial Maurer-Cartan set for the study of the homotopy of mapping spaces in the category of non symmetric operads. For every $n \geq 0$, and for every non-symmetric coaugmented cooperad $\mathcal{C}$ such that $\mathcal{C}(0) = 0$, we construct a twisting derivation ∂^n on $\mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1} N_*(\Delta^n))$ such that

$$B^c(\mathcal{C}) \otimes \Delta^\bullet := (\mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1} N_*(\Delta^\bullet)), \partial^\bullet)$$

is a cosimplicial frame associated to $B^c(\mathcal{C})$. This leads to the following theorem.

Theorem G. *Let* $\mathcal{C}$ *be a coaugmented cooperad and* $\mathcal{P}$ *be an augmented operad such that* $\mathcal{C}(0) = \mathcal{P}(0) = 0$ *and* $\mathcal{C}(1) = \mathcal{P}(1) = \mathbb{K}$*. Then we have an isomorphism of simplicial sets*

$$\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}),\mathcal{P}) \simeq \mathcal{MC}_\bullet(\mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}})).$$

The computation of the connected components and the homotopy groups of $\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}),\mathcal{P})$ can then be achieved by using Theorem E.

In the symmetric context, the twisting derivation ∂^n constructed above on $\mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1} N_*(\Delta^n))$ does not preserve the action of the symmetric group for every $n \geq 2$. We instead consider a Σ_*-cofibrant replacement of $B^c(\mathcal{C})$ given by the map $B^c(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}}) \xrightarrow{\sim} B^c(\mathcal{C})$, where $\mathbf{Sur}_{\mathbb{K}}$ is the surjection cooperad defined in [BCN23, Theorem A.1]. Using that the action of Σ_n on $\overline{\mathcal{C}}(n) \otimes \overline{\mathbf{Sur}}_{\mathbb{K}}(n)$ is free for every $n \geq 1$, we construct a twisting derivation ∂^n on $\mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1} N_*(\Delta^n))$ for every $n \geq 0$ such that

$$B^c(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}}) \otimes \Delta^\bullet := (\mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1} N_*(\Delta^\bullet)), \partial^\bullet)$$

is a cosimplicial frame associated to $B^c(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}})$. We deduce the following theorem.

Theorem H. *Let* $\mathcal{C}$ *be a symmetric coaugmented cooperad and* $\mathcal{P}$ *be a symmetric augmented operad such that* $\mathcal{C}(0) = \mathcal{P}(0) = 0$ *and* $\mathcal{C}(1) = \mathcal{P}(1) = \mathbb{K}$*. Then* $\Sigma\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes N_*(\Delta^\bullet), \overline{\mathcal{P}})$ *is endowed with the structure of a* $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$*-algebra such that we have an isomorphism of simplicial sets*

$$\mathrm{Map}^h_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}),\mathcal{P}) \simeq \mathcal{MC}(\Sigma\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes N_*(\Delta^\bullet), \mathcal{P})),$$

where $\mathrm{Map}^h_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}),\mathcal{P}) := \mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}}),\mathcal{P})$.

Plan of the thesis

This thesis is composed of two chapters written in English:

- Chapter 1, which is a version without modification of the paper *Pre-Lie algebras with divided powers and the Deligne groupoid in positive characteristic* [Ver23] to appear in the journal *Algebraic and Geometric Topology*,
- Chapter 2, which is a follow-up of Chapter 1, which will give the matter of a subsequent memoir,

and one appendix, which provides detailed recollections on operad theory.

General conventions

In this thesis, we deal with differential graded $\mathbb{K}$-modules (dg $\mathbb{K}$-modules for short) over a fixed ground ring $\mathbb{K}$.

In Chapter 1, we adopt cohomological conventions, because we aim to generalize results expressed in this setting in the literature. In this context, a dg $\mathbb{K}$-module is a $\mathbb{K}$-module V endowed with a decomposition into a direct sum of submodules $V \simeq \bigoplus_{n\in\mathbb{Z}} V^n$ and which comes equipped with a differential $d : V \longrightarrow V$ such that $d(V^n) \subset V^{n+1}$ for every $n \in \mathbb{Z}$ and $d^2 = 0$.

In Chapter 2, we rather adopt homological conventions in order to tackle applications in homotopy theory. In this context, a dg $\mathbb{K}$-module is a $\mathbb{K}$-module V endowed with a decomposition into a direct sum of submodules $V \simeq \bigoplus_{n\in\mathbb{Z}} V_n$ and which comes equipped with a differential $d : V \longrightarrow V$ such that $d(V_n) \subset V_{n-1}$ for every $n \in \mathbb{Z}$ and $d^2 = 0$.

Recall that we have an equivalence between these notions. More precisely, let $V \simeq \bigoplus_{n\in\mathbb{Z}} V_n$ be a dg $\mathbb{K}$-module with differential d in homological convention. If we set $V^n := V_{-n}$ for every $n \in \mathbb{Z}$, then $V \simeq \bigoplus_{n\in\mathbb{Z}} V^n$ and $d(V^n) \subset V^{n+1}$ for every $n \in \mathbb{Z}$.

A morphism of dg $\mathbb{K}$-modules is a morphism of $\mathbb{K}$-modules $f : V \longrightarrow W$ which preserves the grading, so that $f(V_n) \subset W_n$ for every $n \in \mathbb{Z}$ (in the homological setting), and the differentials so that $fd = df$. We denote the category of dg $\mathbb{K}$-modules by $\mathrm{dgMod}_{\mathbb{K}}$ with the grading convention (cohomological or homological) fixed by the context.

We use that the category of dg $\mathbb{K}$-modules is equipped with the structure of a symmetric monoidal category. In the homological setting, if $V \simeq \bigoplus_{n\in\mathbb{Z}} V_n$ and $W \simeq \bigoplus_{n\in\mathbb{Z}} W_n$ are dg $\mathbb{K}$-modules, then their tensor product $V \otimes W := V \otimes_{\mathbb{K}} W$ is also a dg $\mathbb{K}$-module with $(V \otimes W)_n \simeq \bigoplus_{p+q=n} V_p \otimes W_q$ for every $n \in \mathbb{Z}$. The differential of $V \otimes W$ is defined by $d(v \otimes w) = d(v) \otimes w + (-1)^p v \otimes d(w)$ for every $v \in V_p$ and $w \in W$. The symmetry operator $\tau : V \otimes W \xrightarrow{\simeq} W \otimes V$ is defined by $\tau(x \otimes y) = (-1)^{pq} y \otimes x$ for every $x \in V_p$ and $y \in W_q$. In what follows, we usually adopt $\pm$ for signs which we deduce from an application of this operator. The general rule (the Koszul sign rule) is that any permutation of factors $xy \longmapsto yx$ in a multiplicative expression in a dg

$\mathbb{K}$-module produces a sign $\pm = (-1)^{pq}$ where p is the degree of x and q is the degree of y.

Chapter 1

Pre-Lie algebras with divided powers and the Deligne groupoid in positive characteristic

The purpose of this chapter is to develop a deformation theory controlled by pre-Lie algebras with divided powers over a ring of positive characteristic. We show that every differential graded pre-Lie algebra with divided powers comes with operations, called weighted braces, which we use to generalize the classical deformation theory controlled by Lie algebras over a field of characteristic 0. Explicitly, we define the Maurer-Cartan set, as well as the gauge group, and prove that there is an action of the gauge group on the Maurer-Cartan set. This new deformation theory moreover admits a Goldman-Millson theorem which remains valid over the integers. As an application, we give the computation of the π_0 of a mapping space $\mathrm{Map}(B^c(\mathcal{C}), \mathcal{P})$ with $\mathcal{C}$ and $\mathcal{P}$ suitable cooperad and operad respectively.

Outline of the current chapter

An important result in deformation theory asserts that every deformation problem over a field of characteristic 0 can be encoded by a differential graded Lie algebra (see

[Lur04] and [Pri10]). More precisely, any deformation problem can be described by a solution of the *Maurer-Cartan equation*:

$$d(x) + \frac{1}{2}[x, x] = 0,$$

in some differential graded Lie algebra. The group obtained by the integration of the differential graded Lie algebra into a Lie group, called the *gauge group*, moreover acts on the Maurer-Cartan set. The orbits of this action give isomorphism classes of deformation problems.

In [DSV16], Dotsenko-Shadrin-Vallette showed that if the differential graded Lie algebra comes from a differential graded pre-Lie algebra, then the Maurer-Cartan equation, the gauge group and its action on the Maurer-Cartan set can be described in terms of pre-Lie operations. A differential graded pre-Lie algebra is a vector space L with a bilinear operation $\star : L \otimes L \longrightarrow L$ such that

$$(x \star y) \star z - x \star (y \star z) = (-1)^{|y||z|}((x \star z) \star y - x \star (z \star y)),$$

and which satisfies the Leibniz rule with respect to the differential. Every differential graded pre-Lie algebra is in particular a differential graded Lie algebra with the graded commutator:

$$[x, y] = x \star y - (-1)^{|x||y|} y \star x.$$

Dotsenko-Shadrin-Vallette showed in particular that given a pre-Lie algebra L, the pre-Lie exponential map $exp : L^0 \longrightarrow (1 + L^0)$ induces an isomorphism between the gauge group and the group $(1 + L^0, \circledcirc, 1)$ with $\circledcirc$ the circular product defined by

$$a \circledcirc (1 + b) = \sum_{n \geq 0} \frac{1}{n!} a\{\underbrace{b, \ldots, b}_{n}\},$$

where $-\{-, \ldots, -\}$ denotes the *symmetric braces* determined by the pre-Lie structure $\star$, starting with $x\{y\} = x \star y$. Then, by writing the Maurer-Cartan equation as a zero-square equation, they prove that the action of the gauge group on the Maurer-Cartan set can be computed in terms of the circular product $\circledcirc$ as

$$e^{\lambda} \cdot \alpha = (e^{\lambda} \star \alpha) \circledcirc e^{-\lambda},$$

allowing us to have an easier way to compute the Deligne groupoid associated to any differential graded pre-Lie algebra over a field of characteristic 0.

The aim of this paper is to develop a deformation theory in positive characteristic which generalizes the deformation theory controlled by pre-Lie algebras over a field of characteristic 0 developed in [DSV16]. Our idea is to use differential graded pre-Lie algebras with divided powers.

The notion of a pre-Lie algebra with divided powers (or $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra) has been studied by Cesaro in [Ces18]. He showed in particular that every pre-Lie algebra with divided powers comes equipped with weighted brace operations $-\{-, \ldots, -\}_{r_1, \ldots, r_n}$, for each collection of integers $r_1, \ldots, r_n \geq 0$, which satisfy similar identities as the quantities

$$x\{y_1,\dots,y_n\}_{r_1,\dots,r_n} = \frac{1}{\prod_i r_i!} x\{\underbrace{y_1,\dots,y_1}_{r_1},\dots,\underbrace{y_n,\dots,y_n}_{r_n}\}$$

in a pre-Lie algebra over a field of characteristic 0 (see [Ces18, Propositions 5.9-5.10] for a precise list of these identities).

Every differential graded pre-Lie algebra with divided powers L comes equipped with analogous weighted brace operations $-\{-,\dots,-\}_{r_1,\dots,r_n}$ which satisfy a graded version of the identities satisfied by weighted braces in the non graded framework. In this context, we have an analogue of the Maurer-Cartan equation:

$$d(x) + x\{x\}_1 = 0.$$

With suitable convergence hypothesis, we also get that the circular product $\circledcirc$ can be written as

$$a \circledcirc (1+b) = \sum_{n\geq 0} a\{b\}_n$$

and gives rise to a group structure on $1+L^0$. This group is called the *gauge group* of L. As in characteristic 0, we also show that this group acts on the Maurer-Cartan set of L.

Theorem A. *Let $\mathbb{K}$ be a ring.*

(i) In any complete differential graded pre-Lie algebra with divided powers L, the circular product $\circledcirc$, defined as above, endows the set $1+L^0$ with a group structure.

(ii) If we denote by d the differential of L, then this group acts on the Maurer-Cartan set via the formula

$$(1+\mu)\cdot\alpha = (\alpha + \mu\{\alpha\}_1 - d(\mu)) \circledcirc (1+\mu)^{\circledcirc -1}.$$

We prove that this new deformation theory satisfies an analogue of the Goldman-Millson theorem given in [GM88, §2.4]. Let $\mathrm{Deligne}(L,A)$ be the Deligne groupoid of the complete dg pre-Lie algebra with divided powers $L\otimes\mathfrak{m}_A$, where L is a dg pre-Lie algebra with divided powers and $\mathfrak{m}_A$ the maximal ideal of a local artinian algebra A over the field of fraction $\mathbf{K}$ of $\mathbb{K}$. We precisely get the following result.

Theorem B. *Let $\mathbb{K}$ be a noetherian integral domain and $\boldsymbol{K}$ its field of fractions. Let L and $\overline{L}$ be two positively graded $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebras. Let $\varphi : L \longrightarrow \overline{L}$ be a morphism of $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebras such that $H^0(\varphi)$ and $H^1(\varphi)$ are isomorphisms and $H^2(\varphi)$ is a monomorphism. Then for every local artinian $\boldsymbol{K}$-algebra A, the induced functor $\varphi_* : \mathrm{Deligne}(L,A) \longrightarrow \mathrm{Deligne}(\overline{L},A)$ is an equivalence of groupoids.*

Other approaches to generalize the usual deformation theory in the positive characteristic framework have been proposed recently in the literature. We have for instance a deformation theory in an associative context, via $\mathcal{A}_\infty$-algebras, which is used to study deformations of group representations (see [MR23a]). Another approach is given by (spectral) partition Lie algebras to get a full generalization of the Lurie-Pridham correspondence in the setting of a field with positive characteristic (see [BCN23; BM23]).

The main motivation for the approach developed in this paper is that operadic deformation problems are expressed in terms of pre-Lie structures. The goal is then to compute the π_0 of a mapping space $\mathrm{Map}(B^c(\mathcal{C}), \mathcal{P})$, where we take any dg operad $\mathcal{P}$ on the target and the operad $B^c(\mathcal{C})$ given by the cobar of a dg coaugmented cooperad $\mathcal{C}$ on the source. Recall simply that $B^c(\mathcal{C})$ defines a cofibrant operad when $\mathcal{C}$ is cofibrant as a symmetric sequence (Σ_*-cofibrant). It is well known that, over a field of characteristic 0, the π_0 of this mapping space is the set of isomorphism classes of the Deligne groupoid of the Lie algebra $\mathrm{Hom}_\Sigma(\overline{\mathcal{C}}, \overline{\mathcal{P}})$. Using the pre-Lie algebra structure of $\mathrm{Hom}_\Sigma(\overline{\mathcal{C}}, \overline{\mathcal{P}})$, this can be seen as a consequence of the computations in [DSV16]. To extend this result, we first show that $\mathrm{Hom}_\Sigma(\overline{\mathcal{C}}, \overline{\mathcal{P}})$ admits a structure of dg pre-Lie algebra with divided powers. Then we get the following statement.

Theorem C. *Let $\mathbb{K}$ be a field. Suppose that $\mathcal{C}$ is a Σ_*-cofibrant coaugmented dg cooperad and $\mathcal{P}$ an augmented dg operad. We then have an isomorphism:*

$$\pi_0(\mathrm{Map}(B^c(\mathcal{C}), \mathcal{P})) \simeq \pi_0\mathrm{Deligne}(\mathrm{Hom}_\Sigma(\overline{\mathcal{C}}, \overline{\mathcal{P}})),$$

where $\pi_0\mathrm{Deligne}(\mathrm{Hom}_\Sigma(\overline{\mathcal{C}}, \overline{\mathcal{P}}))$ denotes the set of isomorphism classes of the Deligne groupoid.

This theorem gives a first step for the calculation of the homotopy groups of a mapping space $\mathrm{Map}(B^c(\mathcal{C}), \mathcal{P})$ over any field.

In the first part of this paper, we recall some definitions and properties on pre-Lie algebras and pre-Lie algebras with divided powers: in §1.1.1 we briefly review the definition of the notion of a pre-Lie algebra and the construction of the corresponding operad; in §1.1.2, we review the definition of a pre-Lie algebra with divided powers and of the weighted brace operations.

In the second part, we develop the deformation theory for differential graded pre-Lie algebras with divided powers: in §1.2.1, we study pre-Lie algebras with divided powers in the dg framework; in §1.2.2, we define the circular product and prove assertion (i) of Theorem A; in §1.2.3, we define the Maurer-Cartan set and prove assertion (ii) of Theorem A; in §1.2.4, we finally prove our analogue of the Goldman-Millson theorem (Theorem B) for this new deformation theory.

We conclude this article with our application of this deformation theory for operadic deformation problems: in §1.3.1, we introduce some basic definitions on symmetric sequences and operads which will be useful to write our formulas; in §1.3.2, we study the structure of differential graded pre-Lie algebra with divided powers of the convolution operad; in §1.3.3, we finally give a proof of Theorem C.

Conventions

We denote the symmetric group on n letters by Σ_n. Recall that a permutation $\sigma \in \Sigma_{r_1+\cdots+r_n}$ is a *shuffle permutation of type* $(r_1, \ldots, r_n)$ if σ preserves the order on the subsets $\{r_1+\cdots+r_i+1 < \cdots < r_1+\cdots+r_{i+1}\}$ of $\{1 < \cdots < r_1+\cdots+r_n\}$. The shuffle permutation σ is *pointed* if we also have $\sigma(1) < \sigma(r_1+1) < \cdots < \sigma(r_1+\cdots+r_{n-1}+1)$. We denote by $Sh(r_1, \ldots, r_n)$ the subset of $\Sigma_{r_1+\cdots+r_n}$ composed of shuffle permutations of type $(r_1, \ldots, r_n)$ and by $Sh_*(r_1, \ldots, r_n)$ the subset of $Sh(r_1, \ldots, r_n)$ composed of pointed shuffle permutations.

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1.1 Recollections on pre-Lie algebras with divided powers

We first recall some definitions and basic properties on pre-Lie algebras and pre-Lie algebras with divided powers. Pre-Lie algebras were introduced in deformation theory by Gerstenhaber in [Ger63], while pre-Lie algebras with divided powers were introduced by Cesaro in [Ces18].

In §1.1.1, we give brief recollections on the notion of a pre-Lie algebra. We will more particularly see pre-Lie algebras as algebras over an operad introduced by Chapoton-Livernet in [CL01], the rooted tree operad, of which we also recall the definition in this subsection.

In §1.1.2, we give recollections on the notion of a pre-Lie algebra with divided powers. These objects can be seen as pre-Lie algebras with some extra operations. We will focus on some of these operations called weighted braces that will mimic the quantities which appear in the definition of the circular product.

1.1.1 Pre-Lie algebras and the rooted tree operad

We will use the following basic definitions.

Definition 1.1.1. *A* pre-Lie algebra *over a ring $\mathbb{K}$ is a $\mathbb{K}$-module L endowed with a bilinear morphism $\star : L \otimes L \longrightarrow L$ such that*

$$(x \star y) \star z - x \star (y \star z) = (x \star z) \star y - x \star (z \star y).$$

The category of pre-Lie algebras is isomorphic to the category of symmetric braces algebras (see [OG08] or [LM05]). The symmetric braces $-\{-, \dots, -\}$ are defined by induction on the length of the brace by

$$\begin{aligned} a\{\} &= a, \\ a\{b_1\} &= a \star b_1, \\ \forall n \geq 1, a\{b_1, \dots, b_n\} &= a\{b_1, \dots, b_{n-1}\}\{b_n\} \\ &\quad - \sum_{i=1}^{n-1} a\{b_1, \dots, b_{i-1}, b_i\{b_n\}, b_{i+1}, \dots, b_{n-1}\}, \end{aligned}$$

for all $a, b_1, \dots, b_n \in L$.

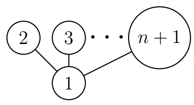
For our purpose, it will be more convenient to see pre-Lie algebras as algebras over an operad. This operad can be described in terms of rooted trees as follows.

Definition 1.1.2 (see [CL01, §1.5]). *We call n-*rooted tree *a non-planar tree with n vertices equipped with a numbering from 1 to n, together with a distinguished vertex called the* root. *By convention, we choose to put the root at the bottom in any representation of a tree.*

We let $\mathcal{RT}(n)$ to be the set of all trees with n vertices, and $\mathcal{P}re\mathcal{L}ie(n) = \mathbb{K}[\mathcal{RT}(n)]$.

The collection $\mathcal{P}re\mathcal{L}ie$ is endowed with an operad structure. The action of Σ_n on $\mathcal{P}re\mathcal{L}ie(n)$ for all $n \geq 1$ is given by the permutation of the indices attached to the vertices. The i-th partial composition $S \circ_i T \in \mathcal{P}re\mathcal{L}ie(p+q-1)$ of $S \in \mathcal{RT}(p)$ and $T \in \mathcal{RT}(q)$ is given by the sum of all the possible trees obtained by putting T in the vertex i of S, with the obvious choice of the numbering (see an example in [CL01, §1.5]). This operad is also called the *rooted tree operad*.

One can show that the algebras over the rooted tree operad are precisely the pre-Lie algebras (see [CL01, §1.9]). In particular, the symmetric braces are given by the trees F_n for $n \geq 0$ called *corollas with n leaves*:

$F_n =$  .

1.1.2 Pre-Lie algebras with divided powers

In this part, we recall the notion of a pre-Lie algebra with divided powers. We obtain this definition as a particular case of a general construction, for algebras over an operad, which we briefly recall.

Every operad $\mathcal{P}$ on a suitable monoidal category C gives a functor $\mathcal{S}(\mathcal{P}, -) : C \longrightarrow C$, called the *Schur functor*, defined by

$$\mathcal{S}(\mathcal{P}, V) = \bigoplus_{n \geq 0} \mathcal{P}(n) \otimes_{\Sigma_n} V^{\otimes n},$$

where we consider, in the direct sum, the coinvariants of $\mathcal{P}(n) \otimes V^{\otimes n}$ under the diagonal action of Σ_n given by its action on $\mathcal{P}(n)$ and its action by permutation on the tensor product $V^{\otimes n}$. The image of a tensor product $p \otimes v_1 \otimes \cdots \otimes v_n \in \mathcal{P}(n) \otimes V^{\otimes n}$ in $\mathcal{P}(n) \otimes_{\Sigma_n} V^{\otimes n}$ will be denoted by $p(v_1, \ldots, v_n)$. The Schur functor defines a monad and the category of algebras over this monad is the usual category of algebras over the operad $\mathcal{P}$. In particular, pre-Lie algebras in the sense of Definition 1.1.1 are identified with $\mathcal{S}(\mathcal{P}re\mathcal{L}ie, -)$-algebras.

In the above definition, one can chose to take invariants instead of coinvariants. We obtain a new functor $\Gamma(\mathcal{P}, -) : C \longrightarrow C$ defined by

$$\Gamma(\mathcal{P}, V) = \bigoplus_{n \geq 0} \mathcal{P}(n) \otimes^{\Sigma_n} V^{\otimes n}.$$

If $\mathcal{P}(0) = 0$, this functor also gives a monad (see [Fre00, §1.1.18]). The algebras over this monad are called *$\mathcal{P}$-algebras with divided powers*. The motivation for this terminology comes from the fact that $\Gamma(\mathcal{C}om, -)$-algebras are precisely the usual commutative and associative algebras over $\mathbb{K}$ with divided powers.

Note that if C is a category whose objects are $\mathbb{K}$-modules, then we have a morphism of monads $Tr : \mathcal{S}(\mathcal{P}, V) \longrightarrow \Gamma(\mathcal{P}, V)$ called the *trace map* and defined by

$$Tr(p(v_1, \ldots, v_n)) = \sum_{\sigma \in \Sigma_n} (\sigma \cdot p) \otimes v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

If $\mathbb{K}$ is a field of characteristic 0, the trace map is an isomorphism. It is no longer the case in general when $char(\mathbb{K}) \neq 0$.

In the case $C = \mathrm{Mod}_{\mathbb{K}}$ of the category of $\mathbb{K}$-modules and $\mathcal{P} = \mathcal{P}re\mathcal{L}ie$, if V is a free $\mathbb{K}$-module, we however have an isomorphism of modules given by the *orbit morphism* $\mathcal{O}$: $\mathcal{S}(\mathcal{P}re\mathcal{L}ie, V) \longrightarrow \Gamma(\mathcal{P}re\mathcal{L}ie, V)$ defined as follows. Let $n \geq 1$ and $\mathfrak{t} \in \mathcal{P}re\mathcal{L}ie(n) \otimes V^{\otimes n}$ be a basis element. We set

$$\mathcal{O}(t) = \sum_{\sigma \in \Sigma_n / \mathrm{Stab}_{\Sigma_n}(t)} \sigma \cdot \mathfrak{t},$$

where $\mathrm{Stab}_{\Sigma_n}(\mathfrak{t})$ is the stabilizer of $\mathfrak{t}$ under the diagonal action of Σ_n on $\mathcal{P}re\mathcal{L}ie(n) \otimes V^{\otimes n}$. The map $\mathcal{O}$ is then extended by linearity on $\mathcal{P}re\mathcal{L}ie(n) \otimes V^{\otimes n}$.

Theorem 1.1.3 (A. Cesaro, [Ces18])**.** *Every pre-Lie algebra with divided powers L comes equipped with operations $-\{-, \ldots, -\}_{r_1, \ldots, r_n} : L^{\times n+1} \longrightarrow L$ called* weighted braces *which satisfy the following identities:*

(i) $x\{y_{\sigma(1)}, \ldots, y_{\sigma(n)}\}_{r_{\sigma(1)}, \ldots, r_{\sigma(n)}} = x\{y_1, \ldots, y_n\}_{r_1, \ldots, r_n}$,

(ii) $x\{y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_n\}_{r_1, \ldots, r_{i-1}, 0, r_{i+1}, \ldots, r_n} = x\{y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n\}_{r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n}$,

(iii) $x\{y_1, \ldots, \lambda y_i, \ldots, y_n\}_{r_1, \ldots, r_i, \ldots, r_n} = \lambda^{r_i} x\{y_1, \ldots, y_i, \ldots, y_n\}_{r_1, \ldots, r_i, \ldots, r_n}$,

(iv) $x\{y_1, \ldots, y_i, y_i, \ldots, y_n\}_{r_1, \ldots, r_i, r_{i+1}, \ldots, r_n} = \binom{r_i + r_{i+1}}{r_i} x\{y_1, \ldots, y_i, \ldots, y_n\}_{r_1, \ldots, r_{i-1}, r_i + r_{i+1}, r_{i+2}, \ldots, r_n}$,

(v) $x\{y_1, \ldots, y_i + \widetilde{y}_i, \ldots, y_n\}_{r_1, \ldots, r_i, \ldots, r_n} = \sum_{s=0}^{r_i} x\{y_1, \ldots, y_i, \widetilde{y}_i, \ldots, y_n\}_{r_1, \ldots, s, r_i - s, \ldots, r_n}$,

(vi) $x\{y_1, \ldots, y_n\}_{r_1, \ldots, r_n}\{z_1, \ldots, z_m\}_{s_1, \ldots, s_m} =$

$$\sum_{s_i = \beta_i + \sum \alpha_i^{\bullet, \bullet}} \frac{1}{\prod_j (r_j)!} x\{y_1\{z_1, \ldots, z_m\}_{\alpha_1^{1,1}, \ldots, \alpha_m^{1,1}}, \ldots, y_1\{z_1, \ldots, z_m\}_{\alpha_1^{1,r_1}, \ldots, \alpha_m^{1,r_1}},$$

$$\ldots, y_n\{z_1, \ldots, z_m\}_{\alpha_1^{n,1}, \ldots, \alpha_m^{n,1}}, \ldots, y_n\{z_1, \ldots, z_m\}_{\alpha_1^{n,r_n}, \ldots, \alpha_m^{n,r_n}}, z_1, \ldots, z_m\}_{1, \ldots, 1, \beta_1, \ldots, \beta_m},$$

for all $n, m \geq 0$, $r_1, \ldots, r_n, s_1, \ldots, s_m \geq 0$, $1 \leq i \leq n$, $\sigma \in \Sigma_n$, $\lambda \in \mathbb{K}$ and $x, y_1, \ldots, y_n, z_1, \ldots, z_m \in L$.

Note that the formula (vi) is written in a form that uses fractions for more convenience, but can be reduced to $\mathbb{Z}$ using the other formulas. The process works as follows. Let i such that $1 \leq i \leq n$. In the sum, we first fix $\beta_1, \ldots, \beta_m$ and $\alpha_j^{p,q}$ for $1 \leq j \leq m$, $1 \leq q \leq r_j$ and $p \neq i$. We obtain a sum with $(\alpha_1^{i,1}, \ldots, \alpha_m^{i,1}, \ldots, \alpha_1^{i,r_i}, \ldots, \alpha_m^{i,r_i})$ as variables. We identify this last tuple with a tuple of tuples of the form $((\alpha_1^{i,1}, \ldots, \alpha_m^{i,1}); \ldots; (\alpha_1^{i,r_i}, \ldots, \alpha_m^{i,r_i}))$.

Let u be one of these tuples and suppose $u = (\underbrace{\widetilde{u_1}, \ldots, \widetilde{u_1}}_{t_1}, \ldots, \underbrace{\widetilde{u_q}, \ldots, \widetilde{u_q}}_{t_q})$ up to permutation. Note that, if $\widetilde{u_1}, \ldots, \widetilde{u_q}$ are given, we exactly have $\dfrac{r_i!}{t_1! \cdots t_q!}$ such terms occurring in the sum. Then, by using the symmetry formula (i), the formula (iv) and by summing over all such tuples, we have in the sum:

$$\frac{1}{\prod_j (r_j)!} \frac{r_i!}{t_1! \cdots t_q!} t_1! \cdots t_q! \; x\{y_1\{z_1, \ldots, z_m\}_{\alpha_1^{1,1}, \ldots, \alpha_m^{1,1}}, \ldots, y_1\{z_1, \ldots, z_m\}_{\alpha_1^{1,r_1}, \ldots, \alpha_m^{1,r_1}}, \ldots,$$

$$y_i\{z_1, \ldots, z_m\}_{\widetilde{u_1}}, \ldots, y_i\{z_1, \ldots, z_m\}_{\widetilde{u_q}}, \ldots$$

$$\ldots, y_n\{z_1, \ldots, z_m\}_{\alpha_1^{n,1}, \ldots, \alpha_m^{n,1}}, \ldots, y_n\{z_1, \ldots, z_m\}_{\alpha_1^{n,r_n}, \ldots, \alpha_m^{n,r_n}}, z_1, \ldots, z_m\}_{1, \ldots, t_1, \ldots, t_q, \ldots, 1, \beta_1, \ldots, \beta_m},$$

where we have set $y_i\{z_1, \ldots, z_m\}_{\widetilde{u_k}} = y_i\{z_1, \ldots, z_m\}_{\alpha_1, \ldots, \alpha_m}$ if $\widetilde{u_k} = (\alpha_1, \ldots, \alpha_m)$. Hence, it gives:

$$\frac{1}{\prod_{j \neq i} (r_j)!} x\{y_1\{z_1, \ldots, z_m\}_{\alpha_1^{1,1}, \ldots, \alpha_m^{1,1}}, \ldots, y_1\{z_1, \ldots, z_m\}_{\alpha_1^{1,r_1}, \ldots, \alpha_m^{1,r_1}}, \ldots,$$

$$y_i\{z_1, \ldots, z_m\}_{\widetilde{u_1}}, \ldots, y_i\{z_1, \ldots, z_m\}_{\widetilde{u_q}}, \ldots$$

$$\ldots, y_n\{z_1, \ldots, z_m\}_{\alpha_1^{n,1}, \ldots, \alpha_m^{n,1}}, \ldots, y_n\{z_1, \ldots, z_m\}_{\alpha_1^{n,r_n}, \ldots, \alpha_m^{n,r_n}}, z_1, \ldots, z_m\}_{1, \ldots, t_1, v, t_q, \ldots, 1, \beta_1, \ldots, \beta_m}.$$

By iterating this argument on the other terms, we obtain an expression over $\mathbb{Z}$.

The reader can find an example of such a reduction of the formula (vi) in [Ces18, Example 5.11], as well as a proof of the previous theorem (see [Ces18, Propositions 5.9-5.10]).

We give the explicit construction of the weighted braces.

Construction 1.1.4. *We regard the weighted braces $x\{y_1, \ldots, y_n\}_{r_1, \ldots, r_n}$ as the action of the corolla $F_{\sum_i r_i}$ on the tensor $x \otimes \underbrace{y_1 \otimes \cdots \otimes y_1}_{r_1} \otimes \cdots \otimes \underbrace{y_n \otimes \cdots \otimes y_n}_{r_n}$ where we regard the y_i's as distinct variables (see below). If $y_i \neq y_j$ for all $i \neq j$, then we precisely set*

$$x\{y_1, \ldots, y_n\}_{r_1, \ldots, r_n} = \gamma(\mathcal{O}F_{\sum_i r_i}(x, \underbrace{y_1, \ldots, y_1}_{r_1}, \ldots, \underbrace{y_n, \ldots, y_n}_{r_n})),$$

where γ is the $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra structure on L.

In order to include the case where some of the y_i's might be the same, let E_n to be the free $\mathbb{K}$-module generated by a basis $e, e_1, \ldots, e_n$. Let $\psi_{x,y_1,\ldots,y_n} : E_n \longrightarrow L$ be the morphism which sends e to x and e_i to y_i for all $1 \leq i \leq n$. We obtain a morphism $\Gamma(\mathcal{P}re\mathcal{L}ie, \psi_{x,y_1,\ldots,y_n}) : \Gamma(\mathcal{P}re\mathcal{L}ie, E_n) \longrightarrow \Gamma(\mathcal{P}re\mathcal{L}ie, L)$. We then take the orbit map at the source and apply this morphism next to have a good definition of the weighted braces.

Remark 1.1.5. *The converse of the previous theorem is also true, provided that L is free as a $\mathbb{K}$-module.*

1.2 Deformation theory of $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebras

The main goal of this section is to extend the results proved by Dotsenko-Shadrin-Vallette in [DSV16] in the context of a ring of positive characteristic. The main idea is that formulas which define the circular product and the gauge action can be written in terms of weighted brace operations.

In §1.2.1, we revisit the definition of pre-Lie algebras with divided powers in the dg framework. In particular, we give the analogue of the weighted brace operations. We then make explicit an example of differential graded pre-Lie algebras with divided powers given by differential graded brace algebras.

In §1.2.2, we define the circular product $\circledcirc$ in terms of weighted brace operations that will generalize the one given in [DSV16]. We then show that this induces a group called the gauge group associated to the $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra.

In §1.2.3, we define the Maurer-Cartan equation in a $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra, and then the Maurer-Cartan set. We also see that the gauge group acts on the Maurer-Cartan set by a similar formula given in [DSV16].

In §1.2.4, we finally motivate this new deformation theory with an analogue of the Goldman-Millson theorem. This theorem, in particular, has the advantage to be true on integers.

1.2.1 Differential graded pre-Lie algebras with divided powers

As we are dealing with differential graded modules, our first goal is to define and study differential graded pre-Lie algebras with divided powers.

In the following sections, we assume that dg modules are equipped with a cohomological grading convention. We will denote by $\otimes$ the usual tensor product of graded modules over any ring $\mathbb{K}$. This induces a symmetric monoidal category that we will denote by $\mathrm{dgMod}_{\mathbb{K}}$. If there is no possible confusion, then we will denote by $\pm$ any sign produced by the Koszul sign rule.

Weighted braces on $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebras

Our main goal here is to extend [Ces18, Proposition 5.13] in the context of dg modules. We first begin by a basic definition.

Definition 1.2.1. *A* differential graded pre-Lie algebra *is an algebra over the monad* $\mathcal{S}(\mathcal{P}re\mathcal{L}ie, -) : \mathrm{dgMod}_{\mathbb{K}} \longrightarrow \mathrm{dgMod}_{\mathbb{K}}$.

Equivalently, we can see that a differential graded pre-Lie algebra is a graded module $L = \bigoplus_{k\in\mathbb{Z}} L^k$ endowed with a morphism of graded modules $\star : L \otimes L \longrightarrow L$ such that

$$(x \star y) \star z - x \star (y \star z) = \pm((x \star z) \star y - x \star (z \star y))$$

and a differential $d : L^k \longrightarrow L^{k+1}$, which satisfies

$$d(x \star y) = d(x) \star y \pm x \star d(y),$$

where $\pm$ is the sign yielded by the permutation of x and d.

We now define the notion of a pre-Lie algebra with divided powers in the dg framework.

Definition 1.2.2. *A* differential graded pre-Lie algebra with divided powers *is an algebra over the monad* $\Gamma(\mathcal{P}re\mathcal{L}ie, -) : \mathrm{dgMod}_{\mathbb{K}} \longrightarrow \mathrm{dgMod}_{\mathbb{K}}$.

Let $L \in \mathrm{dgMod}_{\mathbb{K}}$. Suppose that L^k is a free $\mathbb{K}$-module for every $k \in \mathbb{Z}$. Let $\mathcal{L}$ be a basis of L composed of homogeneous elements. Then we have a basis on $\mathcal{P}re\mathcal{L}ie(n) \otimes L^{\otimes n}$ for every $n \geq 1$ given by tensors $T \otimes e_1 \otimes \cdots \otimes e_n$ where $T \in \mathcal{RT}(n)$ and $e_1, \ldots, e_n \in \mathcal{L}$. We denote by $\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n}$ this basis. If $char(\mathbb{K}) \neq 2$, then the action of Σ_n on $\mathcal{P}re\mathcal{L}ie(n) \otimes L^{\otimes n}$ does not restrict to an action on $\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n}$ because of the Koszul sign rule. To handle things properly, we put this sign apart. We endow $\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n}$ with the diagonal action of Σ_n where Σ_n acts on $\mathcal{L}^{\otimes n}$ by the permutation of elements where we omit the Koszul sign. Given a tensor $\mathfrak{t} \in \mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n}$, we denote by $X_{\mathfrak{t}}$ the orbit of $\mathfrak{t}$ under this action so that we have the following equality of graded $\mathbb{K}$-modules:

$$\mathcal{P}re\mathcal{L}ie(n) \otimes L^{\otimes n} = \bigoplus_{\bar{\mathfrak{t}} \in (\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n})/\Sigma_n} \mathbb{K}[X_{\mathfrak{t}}].$$

We recover the Koszul sign rule in the following way. Let $\mathfrak{t} \in \mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n}$. For every $\sigma \in \Sigma_n$, we define $\varepsilon(\sigma, \mathfrak{t}) \in \{\pm 1\} \subset \mathbb{K}^{\times}$ as the Koszul sign which appears after the usual action of σ on $\mathfrak{t}$ in the graded module $\mathcal{P}re\mathcal{L}ie(n) \otimes L^{\otimes n}$. For every $\sigma, \tau \in \Sigma_n$, we have the identity

$$\varepsilon(\sigma\tau, \mathfrak{t}) = \varepsilon(\sigma, \tau \cdot \mathfrak{t})\varepsilon(\tau, \mathfrak{t}).$$

Equivalently, we can see ε as a functor from the groupoid with $X_{\mathfrak{t}}$ as set of objects and $\mathrm{Hom}(\mathfrak{t}', \mathfrak{t}'') = \{\sigma \in \Sigma_n \mid \sigma \cdot \mathfrak{t}' = \mathfrak{t}''\}$ for $\mathfrak{t}', \mathfrak{t}'' \in X_{\mathfrak{t}}$ to the groupoid, denoted by $\{\pm 1\}$, with only one object $*$ and $\mathrm{Hom}(*, *) = \{\pm 1\} \subset \mathbb{K}^{\times}$. We can then define the Σ_n-representation $\mathbb{K}[X_{\mathfrak{t}}]^{\pm}$ as $\mathbb{K}[X_{\mathfrak{t}}]$ endowed with the action of Σ_n defined by

$$\sigma \cdot x^{\pm} = \varepsilon(\sigma, x)(\sigma \cdot x)^{\pm}$$

for every $\sigma \in \Sigma_n$ and $x \in X_{\mathfrak{t}}$. We thus have the following identity of Σ_n-representations:

$$\mathcal{P}re\mathcal{L}ie(n) \otimes L^{\otimes n} = \bigoplus_{\bar{\mathfrak{t}} \in (\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n})/\Sigma_n} \mathbb{K}[X_{\mathfrak{t}}]^{\pm}.$$

Our purpose it to define an analogue of the orbit map, by using the above decomposition in set-theoretic orbits. We rely on the following lemma.

Lemma 1.2.3. *Let G be a group and $H \subset G$ be a subgroup. We consider the action of G on G/H by the left translation. Let X be the groupoid with as set of objects G/H and with as morphisms* $\mathrm{Hom}(x, x') = \{g \in G \mid g \cdot x = x'\}$. *Let* $\varepsilon : X \longrightarrow \{\pm 1\} \subset \mathbb{K}^{\times}$

be a functor. We denote by $\varepsilon(g,x)$ *the image of the morphism* $g : x \longrightarrow g\cdot x$ *under this functor. Consider the* G*-representation* $\mathbb{K}[G/H]^{\pm} = \mathbb{K}[G/H]$ *on which* G *acts by* $g\cdot x^{\pm} = \varepsilon(g,x)(g\cdot x)^{\pm}$ *for every* $x \in G/H$*. For every* $g \in G$*, we denote by* $\overline{g}$ *its class in* G/H *and by* $[\overline{g}^{\pm}]$ *the class of* $\overline{g}^{\pm} \in \mathbb{K}[G/H]^{\pm}$ *in* $(\mathbb{K}[G/H]^{\pm})_G$*.*

— *If there exists* $h \in H$ *such that* $\varepsilon(h,\overline{1}) \neq 1$*, then*

$$(\mathbb{K}[G/H]^{\pm})_G = (\mathbb{K}/2\mathbb{K})[[\overline{1}^{\pm}]]\ ;\ (\mathbb{K}[G/H]^{\pm})^G = \mathrm{Tor}_2(\mathbb{K})\left[\sum_{\overline{g}\in G/H}\varepsilon(g,\overline{1})\overline{g}^{\pm}\right],$$

where $\mathrm{Tor}_2(\mathbb{K})$ *denotes the set of* 2*-torsion elements of* $\mathbb{K}$*.*

— *Otherwise, we have the identities*

$$(\mathbb{K}[G/H]^{\pm})_G = \mathbb{K}[[\overline{1}^{\pm}]]\ ;\ (\mathbb{K}[G/H]^{\pm})^G = \mathbb{K}\left[\sum_{\overline{g}\in G/H}\varepsilon(g,\overline{1})\overline{g}^{\pm}\right].$$

Proof. We first compute $(\mathbb{K}[G/H]^{\pm})_G$. For every $g \in G$, we have $\overline{g}^{\pm} = g\cdot(\varepsilon(g,\overline{1})\overline{1}^{\pm})$ so that the $\mathbb{K}$-module $(\mathbb{K}[G/H]^{\pm})_G$ is generated by $[\overline{1}^{\pm}]$. If there exists $h \in H$ such that $\varepsilon(h,\overline{1}) \neq 1$ (meaning that $\varepsilon(h,\overline{1}) = -1 \neq 1$ in $\mathbb{K}$), then, for every $\lambda \in \mathbb{K}$, we have that $\lambda[\overline{1}^{\pm}] = \lambda[h\cdot\overline{1}^{\pm}] = -\lambda[\overline{1}^{\pm}]$ which shows that $(\mathbb{K}[G/H]^{\pm})_G = (\mathbb{K}/2\mathbb{K})[[\overline{1}^{\pm}]]$. If $\varepsilon(H,\overline{1}) \subset \{1\}$, then $(\mathbb{K}[G/H]^{\pm})_G = \mathbb{K}[[\overline{1}^{\pm}]]$.

We now compute $(\mathbb{K}[G/H]^{\pm})^G$. Let $x = \sum_{\overline{g}\in G/H}\lambda_{\overline{g}}\overline{g}^{\pm} \in (\mathbb{K}[G/H]^{\pm})^G$. For every $g \in G$, the identity $g\cdot x = x$ gives $\lambda_{\overline{g}} = \varepsilon(g,\overline{1})\lambda_{\overline{1}}$. If there exists $h \in H$ such that $\varepsilon(h,\overline{1}) \neq 1$, then $\varepsilon(g,\overline{1})\lambda_{\overline{1}} = \lambda_{\overline{g}} = \lambda_{\overline{gh}} = \varepsilon(gh,\overline{1})\lambda_{\overline{1}} = \varepsilon(g,\overline{1})\varepsilon(h,\overline{1})\lambda_{\overline{1}}$ which gives $2\lambda_{\overline{1}} = 0$. We then have $\lambda_{\overline{1}} \in \mathrm{Tor}_2(\mathbb{K})$ which shows that $(\mathbb{K}[G/H]^{\pm})^G = \mathrm{Tor}_2(\mathbb{K})[\sum_{\overline{g}\in G/H}\varepsilon(g,\overline{1})\overline{g}^{\pm}]$. If $\varepsilon(H,\overline{1}) \subset \{1\}$ then $(\mathbb{K}[G/H]^{\pm})^G = \mathbb{K}[\sum_{\overline{g}\in G/H}\varepsilon(g,\overline{1})\overline{g}^{\pm}]$. □

Lemma 1.2.4. *In the situation of Lemma 1.2.3 and when* $\varepsilon(H,\overline{1}) \subset \{1\}$*, we define the* orbit map $\mathcal{O} : (\mathbb{K}[G/H]^{\pm})_G \longrightarrow (\mathbb{K}[G/H]^{\pm})^G$ *by*

$$\mathcal{O}([\overline{1}]^{\pm}) = \sum_{\overline{g}\in G/H}\varepsilon(g,\overline{1})\overline{g}^{\pm}.$$

This map is an isomorphism.

Proof. This is an immediate consequence of the second point of Lemma 1.2.3. □

We can apply this lemma to our situation, by noting that for every $\mathfrak{t} \in \mathcal{RT}(n)\otimes\mathcal{L}^{\otimes n}$, the set $X_{\mathfrak{t}}$ is in bijection with $\Sigma_n/\mathrm{Stab}_{\Sigma_n}(\mathfrak{t})$. In order to apply Lemma 1.2.4, we need to remove some elements of $\mathcal{RT}(n)\otimes\mathcal{L}^{\otimes n}$. These elements are given by tensors $\mathfrak{t} \in \mathcal{RT}(n)\otimes\mathcal{L}^{\otimes n}$ such that there exists $\sigma \in \mathrm{Stab}_{\Sigma_n}(\mathfrak{t})$ with $\varepsilon(\sigma,\mathfrak{t}) \neq 1$. We denote by $(\mathcal{RT}(n)\otimes\mathcal{L}^{\otimes n})^o$ the set of such elements and $(\mathcal{RT}(n)\otimes\mathcal{L}^{\otimes n})^r = (\mathcal{RT}(n)\otimes\mathcal{L}^{\otimes n}) \setminus (\mathcal{RT}(n)\otimes\mathcal{L}^{\otimes n})^o$. In the case $char(\mathbb{K}) = 2$, we just have $(\mathcal{RT}(n)\otimes\mathcal{L}^{\otimes n})^r = \mathcal{RT}(n)\otimes\mathcal{L}^{\otimes n}$.

We note that these sets are stable under the action of Σ_n. Indeed, if $\mathfrak{t} \in \mathcal{RT}(n)\otimes\mathcal{L}^{\otimes n}$ is such that there exists $\sigma \in \mathrm{Stab}_{\Sigma_n}(\mathfrak{t})$ with $\varepsilon(\sigma,\mathfrak{t}) \neq 1$, then for every $\tau \in \Sigma_n$, we have that $\tau\sigma\tau^{-1} \in \mathrm{Stab}_{\Sigma_n}(\tau\cdot\mathfrak{t})$ and

$$\varepsilon(\tau\sigma\tau^{-1},\tau\cdot\mathfrak{t}) = \varepsilon(\tau,\sigma\cdot\mathfrak{t})\varepsilon(\sigma,\mathfrak{t})\varepsilon(\tau^{-1},\tau\cdot\mathfrak{t}) = \varepsilon(\tau,\mathfrak{t})\varepsilon(\sigma,\mathfrak{t})\varepsilon(\tau^{-1},\tau\cdot\mathfrak{t}) = \varepsilon(\sigma,\mathfrak{t}) \neq 1.$$

We deduce that the quotient $(\mathcal{RT}(n)\otimes\mathcal{L}^{\otimes n})/\Sigma_n$ is the disjoint union of $(\mathcal{RT}(n)\otimes\mathcal{L}^{\otimes n})^o/\Sigma_n$ and $(\mathcal{RT}(n)\otimes\mathcal{L}^{\otimes n})^r/\Sigma_n$. We then define

$$\mathcal{S}^r(\mathcal{P}re\mathcal{L}ie,L) = \bigoplus_{n\geq 1}(\mathbb{K}[(\mathcal{RT}(n)\otimes\mathcal{L}^{\otimes n})^r]^{\pm})_{\Sigma_n} \subset \mathcal{S}(\mathcal{P}re\mathcal{L}ie,L).$$

We deduce the following proposition.

Proposition 1.2.5. *If $\mathbb{K}$ is an integral domain and if L^k is a free $\mathbb{K}$-module for every $k\in\mathbb{Z}$, then the map $\mathcal{O}:\mathcal{S}^r(\mathcal{P}re\mathcal{L}ie,L)\longrightarrow\Gamma(\mathcal{P}re\mathcal{L}ie,L)$ is an isomorphism.*

Proof. Let $\mathcal{L}$ be a basis of L composed of homogeneous elements. We adopt the same notations as before Lemma 1.2.3 and before the statement of Proposition 1.2.5. We then note that we have

$$\mathcal{S}^r(\mathcal{P}re\mathcal{L}ie,L) = \bigoplus_{n\geq 1}\bigoplus_{\bar{\mathfrak{t}}\in(\mathcal{RT}(n)\otimes\mathcal{L}^{\otimes n})^r/\Sigma_n}(\mathbb{K}[X_{\mathfrak{t}}]^{\pm})_{\Sigma_n};$$

$$\Gamma(\mathcal{P}re\mathcal{L}ie,L) = \bigoplus_{n\geq 1}\bigoplus_{\bar{\mathfrak{t}}\in(\mathcal{RT}(n)\otimes\mathcal{L}^{\otimes n})/\Sigma_n}(\mathbb{K}[X_{\mathfrak{t}}]^{\pm})^{\Sigma_n}.$$

If $char(\mathbb{K})=2$, then $2\mathbb{K}=0$ and $\mathrm{Tor}_2(\mathbb{K})=\mathbb{K}$ so that the proposition is an immediate consequence of Lemma 1.2.4.
If $char(\mathbb{K})\neq 2$, then by the first point of Lemma 1.2.3 and by noting that $\mathrm{Tor}_2(\mathbb{K})=0$ because $\mathbb{K}$ is an integral domain, we have that $(\mathbb{K}[(\mathcal{RT}(n)\otimes\mathcal{L}^{\otimes n})^o]^{\pm})^{\Sigma_n}=0$. We are then reduced to analyze the orbit maps $\mathcal{O}:(\mathbb{K}[X_{\mathfrak{t}}]^{\pm})_{\Sigma_n}\longrightarrow(\mathbb{K}[X_{\mathfrak{t}}]^{\pm})^{\Sigma_n}$ for $\mathfrak{t}\in(\mathcal{RT}(n)\otimes\mathcal{L}^{\otimes n})^r$, which are isomorphisms by the second point of Lemma 1.2.4.

In any case, we then obtain that $\mathcal{O}:\mathcal{S}^r(\mathcal{P}re\mathcal{L}ie,L)\longrightarrow\Gamma(\mathcal{P}re\mathcal{L}ie,L)$ is an isomorphism. □

Theorem 1.2.6. *Let $\mathbb{K}$ be a ring. A graded pre-Lie algebra with divided powers $L=\bigoplus_{k\in\mathbb{Z}}L^k$ over $\mathbb{K}$ comes equipped with operations, called* weighted braces, *which have the following form.*

- *If $char(\mathbb{K})=2$, weighted braces are maps*

$$-\{-,\dots,-\}_{r_1,\dots,r_n}:L^{\times n+1}\longrightarrow L,$$

defined for any collection of integers $r_1,\dots,r_n\geq 0$, which satisfy all formulas of Theorem 1.1.3 and preserve the grading in the sense that

$$L^k\{L^{k_1},\dots,L^{k_n}\}_{r_1,\dots,r_n}\subset L^{k+k_1r_1+\dots+k_nr_n}.$$

- *If $char(\mathbb{K})\neq 2$, by setting $L^{ev}=\bigoplus_{k\in\mathbb{Z}}L^{2k}$ and $L^{odd}=\bigoplus_{k\in\mathbb{Z}}L^{2k+1}$, weighted* *braces are maps*

$$-\{\underbrace{-,\ldots,-}_{p},\underbrace{-,\ldots,-}_{q}\}_{r_1,\ldots,r_p,1,\ldots,1} : L\times (L^{ev})^{\times p}\times (L^{odd})^{\times q}\longrightarrow L,$$

defined for any collection of integers $p,q,r_1,\ldots,r_p\geq 0$, which satisfy all formulas of Theorem 1.1.3 with a sign given by the Koszul sign rule (see the Remark 1.2.7 below) and preserve the grading.

Conversely, if $\mathbb{K}$ is an integral domain, if L^k is a free $\mathbb{K}$-module for every $k\in\mathbb{Z}$ and if L admits weighted brace operations, then L is a $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra.

Remark 1.2.7. *If $char(\mathbb{K})\neq 2$, formulas (i) and (vi) of Theorem 1.1.3 differ by a sign given by the Koszul sign rule. The sign which appears in formula (i) is given by*

$$y_{\sigma(1)}^{\otimes r_{\sigma(1)}}\otimes\cdots\otimes y_{\sigma(n)}^{\otimes r_{\sigma(n)}}\longmapsto \pm y_1^{\otimes r_1}\otimes\cdots\otimes y_n^{\otimes r_n}.$$

In formula (vi), for each β_i's and $\alpha_i^{\bullet,\bullet}$'s, the sign which appears in the relevant term is given by

$$z_1^{\otimes s_1}\otimes\cdots\otimes z_m^{\otimes s_m}\longmapsto \pm z_1^{\otimes\alpha_1^{1,1}}\otimes\cdots\otimes z_m^{\otimes\alpha_m^{1,1}}\otimes\cdots\otimes z_1^{\otimes\alpha_1^{n,1}}\otimes\cdots\otimes z_m^{\otimes\alpha_m^{n,r_n}}\otimes z_1^{\otimes\beta_1}\otimes\cdots\otimes z_m^{\otimes\beta_m}.$$

Note that these signs are induced by the permutation of the odd degree elements between them. Since their associated weight are equal to 1, formula (vi) can still be written without rational coefficients by the same process as in the paragraph after Theorem 1.1.3.

In order to handle both of the cases, in the following, when taking elements with associated weights, we will tacitly suppose that if $char(\mathbb{K})\neq 2$, then all odd degree elements will have an associated weight equal to 1.

Proof. We basically do the same thing as in [Ces18, Proposition 5.10]. Let $x,y_1,\ldots,y_n\in L$. Let $E_{x,y_1,\ldots,y_n}$ be the graded $\mathbb{K}$-module generated by $e,e_1,\ldots,e_n$ with matching degrees. We have a morphism of graded modules from $\psi_{x,y_1,\ldots,y_n}: E_{x,y_1,\ldots,y_n}\longrightarrow L$ which sends e to x and e_i to y_i for every $1\leq i\leq n$. This gives rise by functoriality to a morphism $\Gamma(\mathcal{P}re\mathcal{L}ie,\psi_{x,y_1,\ldots,y_n}):\Gamma(\mathcal{P}re\mathcal{L}ie,E_{x,y_1,\ldots,y_n})\longrightarrow\Gamma(\mathcal{P}re\mathcal{L}ie,L)$. We set

$$x\{y_1,\ldots,y_n\}_{r_1,\ldots,r_n}:=l(\Gamma(\mathcal{P}re\mathcal{L}ie,\psi_{x,y_1,\ldots,y_n})(\mathcal{O}F_{\sum_i r_i}(e,\underbrace{e_1,\ldots,e_1}_{r_1},\ldots,\underbrace{e_n,\ldots,e_n}_{r_n}))),$$

where l is the $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra structure on L. One can check that all the desired formulas are satisfied.

Now suppose that $\mathbb{K}$ is an integral domain and that L admits weighted brace operations $-\{-,\ldots,-\}_{r_1,\ldots,r_n}$. We first note that every elements in $\Gamma(\mathcal{P}re\mathcal{L}ie,L)$ can be described as a sum of iterated monadic compositions of corollas in some basis of homogeneous elements of L. This can be proved by using Proposition 1.2.5 and by following the same proofs of [Ces18, Theorem 5.1] and [Ces18, Lemma 5.2], which come from the computation of the monadic composition in $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$. We next pick elements $x, y_1, \ldots, y_n$ of the chosen basis such that $y_i \neq y_j$ if $i \neq j$, and set

$$l(\mathcal{O}F_{\sum_i r_i}(x, \underbrace{y_1, \ldots, y_1}_{r_1}, \ldots, \underbrace{y_n, \ldots, y_n}_{r_n})) := x\{y_1, \ldots, y_n\}_{r_1,\ldots,r_n}.$$

Since every elements of $\Gamma(\mathcal{P}re\mathcal{L}ie, L)$ can be described as a composite of corollas in $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$, we have defined $l : \Gamma(\mathcal{P}re\mathcal{L}ie, L) \longrightarrow L$. We see that this construction does not depend on the choice of the basis of the L^k's. Indeed, we can apply the same arguments as in [Ces18, Lemma 5.15], which only rely on computations and on the relations satisfied by weighted braces. The same proof of [Ces18, Lemma 5.18] can also be applied to prove that the resulting morphism l endows L with a structure of a $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra. □

This theorem admits an analogue in the differential graded case.

Theorem 1.2.8. *Every differential graded pre-Lie algebras with divided powers $L = \bigoplus_{k\in\mathbb{Z}} L^k$ admits weighted braces which satisfy the same formulas as in Theorem 1.2.6, and which satisfy in addition the identity*

$$\begin{aligned} d(x\{y_1, \ldots, y_n\}_{r_1,\ldots,r_n}) = {} & d(x)\{y_1, \ldots, y_n\}_{r_1,\ldots,r_n} \\ & + \sum_{k=1}^{n} (-1)^{\varepsilon_k} x\{y_1, \ldots, y_k, d(y_k), \ldots, y_n\}_{r_1,\ldots,r_k-1,1,\ldots,r_n}, \end{aligned}$$

where $\varepsilon_k = |x| + |y_1| + \cdots + |y_{k-1}|$.

Conversely, if $\mathbb{K}$ is an integral domain, if L^k is a free $\mathbb{K}$-module for every $k \in \mathbb{Z}$ and if L admits weighted brace operations which satisfy the previous identity, then L is a $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra.

Proof. Let $x, y_1, \ldots, y_n \in L$. We set $E_{x,y_1,\ldots,y_n}$ to be the dg module generated by elements $e, e_1, \ldots, e_n, f, f_1, \ldots, f_n$ such that $|e| = |x|, |e_1| = |y_1|, \ldots, |e_n| = |y_n|$ and $d(e) = f$, $d(e_1) = f_1, \ldots, d(e_n) = f_n$. We then have a morphism $\psi_{x,y_1,\ldots,y_n} : E_{x,y_1,\ldots,y_n} \longrightarrow L$ of dg modules defined by sending e to x, and e_i (resp. f_i) to y_i (resp. $d(y_i)$) for every $1 \leq i \leq n$. We set

$$x\{y_1, \ldots, y_n\}_{r_1,\ldots,r_n} := l(\Gamma(\mathcal{P}re\mathcal{L}ie, \psi_{x,y_1,\ldots,y_n})(\mathcal{O}F_{\sum_i r_i}(e, \underbrace{e_1, \ldots, e_1}_{r_1}, \ldots, \underbrace{e_n, \ldots, e_n}_{r_n}))),$$

where l is the $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra structure on L. By forgetting the differentials and by applying Theorem 1.2.6, we have that the operations $-\{-, \ldots, -\}_{r_1,\ldots,r_n}$ satisfy all the formulas of Theorem 1.1.3, with a sign. It only remains to prove the compatibility with the differential d.

$$\begin{aligned} d(x\{y_1, \ldots, y_n\}_{r_1,\ldots,r_n}) &= dl(\Gamma(\mathcal{P}re\mathcal{L}ie, \psi_{x,y_1,\ldots,y_n})(\mathcal{O}F_{\sum_i r_i}(e, \underbrace{e_1, \ldots, e_1}_{r_1}, \ldots, \underbrace{e_n, \ldots, e_n}_{r_n}))) \\ &= l(\Gamma(\mathcal{P}re\mathcal{L}ie, \psi_{x,y_1,\ldots,y_n})(d\mathcal{O}F_{\sum_i r_i}(e, \underbrace{e_1, \ldots, e_1}_{r_1}, \ldots, \underbrace{e_n, \ldots, e_n}_{r_n}))), \end{aligned}$$

by commutation of d with the algebra structure l and $\psi_{x,y_1,\ldots,y_n}$. Next, we claim that

$$d\mathcal{O}F_{\sum_i r_i}(e,\underbrace{e_1,\ldots,e_1}_{r_1},\ldots,\underbrace{e_n,\ldots,e_n}_{r_n}) = \mathcal{O}F_{\sum_i r_i}(f,\underbrace{e_1,\ldots,e_1}_{r_1},\ldots,\underbrace{e_n,\ldots,e_n}_{r_n}) + \sum_{k=1}^{n} \pm\mathcal{O}F_{\sum_i r_i}(e,\underbrace{e_1,\ldots,e_1}_{r_1},\ldots,\underbrace{e_k,\ldots,e_k}_{r_k-1},f_k,\ldots,\underbrace{e_n,\ldots,e_n}_{r_n}).$$

Indeed, recall that

$$\mathcal{O}F_{\sum_i r_i}(e,\underbrace{e_1,\ldots,e_1}_{r_1},\ldots,\underbrace{e_n,\ldots,e_n}_{r_n}) = \sum_{\sigma\in Sh(1,r_1,\ldots,r_n)} \sigma\cdot(F_{\sum_i r_i}\otimes e\otimes e_1^{\otimes r_1}\otimes\cdots\otimes e_n^{\otimes r_n}).$$

We then have

$$d\mathcal{O}F_{\sum_i r_i}(e,\underbrace{e_1,\ldots,e_1}_{r_1},\ldots,\underbrace{e_n,\ldots,e_n}_{r_n}) = \mathcal{O}F_{\sum_i r_i}(f,\underbrace{e_1,\ldots,e_1}_{r_1},\ldots,\underbrace{e_n,\ldots,e_n}_{r_n}) + \sum_{k=1}^{n}\sum_{\sigma\in Sh(1,r_1,\ldots,r_n)}\sum_{i=1}^{r_k} \pm\sigma\cdot(F_{\sum_i r_i}\otimes e\otimes e_1^{\otimes r_1}\otimes\cdots\otimes e_k^{\otimes i-1}\otimes f_k\otimes e_k^{r_k-i}\otimes\cdots\otimes e_n^{\otimes r_n}).$$

Let $1\leq k\leq n$. For every $1\leq i\leq r_k$, we define $\tau_{k,i}$ as the permutation which permutes f_k with the block $e_k^{\otimes r_k-i}$. We then obtain

$$d\mathcal{O}F_{\sum_i r_i}(e,\underbrace{e_1,\ldots,e_1}_{r_1},\ldots,\underbrace{e_n,\ldots,e_n}_{r_n}) = \mathcal{O}F_{\sum_i r_i}(f,\underbrace{e_1,\ldots,e_1}_{r_1},\ldots,\underbrace{e_n,\ldots,e_n}_{r_n}) + \sum_{k=1}^{n}\sum_{\sigma\in Sh(1,r_1,\ldots,r_n)}\sum_{i=1}^{r_k} \pm\sigma\tau_{k,i}^{-1}\cdot(F_{\sum_i r_i}\otimes e\otimes e_1^{\otimes r_1}\otimes\cdots\otimes e_k^{\otimes r_k-1}\otimes f_k\otimes\cdots\otimes e_n^{\otimes r_n}).$$

We note that, for every $1\leq i\leq r_k$, the permutation $\sigma\tau_{k,i}^{-1}$ is in $Sh(1,r_1,\ldots,r_k-1,1,\ldots,r_n)$. In the converse direction, if $\widetilde{\sigma}\in Sh(1,r_1,\ldots,r_k-1,1,\ldots,r_n)$, then there is a unique $1\leq i\leq r_k$ such that $\widetilde{\sigma}\tau_{k,i}\in Sh(1,r_1,\ldots,r_n)$. We thus have proved that

$$d\mathcal{O}F_{\sum_i r_i}(e,\underbrace{e_1,\ldots,e_1}_{r_1},\ldots,\underbrace{e_n,\ldots,e_n}_{r_n}) = \mathcal{O}F_{\sum_i r_i}(f,\underbrace{e_1,\ldots,e_1}_{r_1},\ldots,\underbrace{e_n,\ldots,e_n}_{r_n}) + \sum_{k=1}^{n}\sum_{\widetilde{\sigma}\in Sh(1,r_1,\ldots,r_k-1,1,\ldots,r_n)} \pm\widetilde{\sigma}\cdot(F_{\sum_i r_i}\otimes e\otimes e_1^{\otimes r_1}\otimes\cdots\otimes e_k^{\otimes r_k-1}\otimes f_k\otimes\cdots\otimes e_n^{\otimes r_n})$$

which gives

$$d\mathcal{O}F_{\sum_i r_i}(e,\underbrace{e_1,\ldots,e_1}_{r_1},\ldots,\underbrace{e_n,\ldots,e_n}_{r_n}) = \mathcal{O}F_{\sum_i r_i}(f,\underbrace{e_1,\ldots,e_1}_{r_1},\ldots,\underbrace{e_n,\ldots,e_n}_{r_n}) + \sum_{k=1}^{n} \pm\mathcal{O}F_{\sum_i r_i}(e,\underbrace{e_1,\ldots,e_1}_{r_1},\ldots,\underbrace{e_k,\ldots,e_k}_{r_k-1},f_k,\ldots,\underbrace{e_n,\ldots,e_n}_{r_n}).$$

Applying $\Gamma(\mathcal{P}re\mathcal{L}ie,\psi_{x,y_1,\ldots,y_n})$ and l will then give the desired quantity, by definition of the weighted braces.

Suppose now that $\mathbb{K}$ is an integral domain, that L^k is free for every $k \in \mathbb{Z}$ and that L is endowed with weighted brace operations. By Theorem 1.2.6, and by forgetting the differential of L, we can define a morphism of graded modules $l : \Gamma(\mathcal{P}re\mathcal{L}ie, L) \longrightarrow L$ which is compatible with the monadic structure of $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$. We now prove that l commutes with the differential d. Since d commutes with the monadic structure, it is sufficient to prove the commutation with d when reducing to corollas. Let $x, y_1, \dots, y_n$ be some basis elements with $y_i \neq y_j$ if $i \neq j$ and $r_1, \dots, r_n \geq 1$. We then have

$$dl(\mathcal{O}F_{\sum_i r_i}(x, \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n})) = d(x\{y_1, \dots, y_n\}_{r_1,\dots,r_n})$$

$$= d(x)\{y_1, \dots, y_n\}_{r_1,\dots,r_n} + \sum_{k=1}^{n} \pm x\{y_1, \dots, y_k, d(y_k), \dots, y_n\}_{r_1,\dots,r_k-1,1,\dots,r_n}.$$

We decompose $d(y_k)$ in the chosen basis, and write

$$d(y_k) = \sum_{\substack{i=1 \\ i\neq k}}^{n} \lambda_{k,i} y_i + f_k,$$

where $f_k = 0$ or $f_k \notin Vect(y_1, \dots, y_n)$ (note that y_k cannot appear in the decomposition of $d(y_k)$ for degree reason). This gives

$$\begin{aligned} dl(\mathcal{O}F_{\sum_i r_i}(x, \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n})) &= d(x\{y_1, \dots, y_n\}_{r_1,\dots,r_n}) \\ &= d(x)\{y_1, \dots, y_n\}_{r_1,\dots,r_n} \\ + \sum_{k=1}^{n} \sum_{\substack{i=1 \\ i\neq k}}^{n} \pm \lambda_{k,i} x\{y_1, \dots, y_k, y_i, \dots, y_n\}_{r_1,\dots,r_k-1,1,\dots,r_n} \\ + \sum_{k=1}^{n} \pm x\{y_1, \dots, y_k, f_k, \dots, y_n\}_{r_1,\dots,r_k-1,1,\dots,r_n} \end{aligned}$$

and then, by using the symmetry relations,

$$dl(\mathcal{O}F_{\sum_i r_i}(x, \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n})) = d(x\{y_1, \dots, y_n\}_{r_1,\dots,r_n})$$

$$= d(x)\{y_1, \dots, y_n\}_{r_1,\dots,r_n} + \sum_{k=1}^{n} \sum_{\substack{i=1 \\ i\neq k}}^{n} \pm \lambda_{k,i} x\{y_1, \dots, y_i, y_i, \dots, y_k, \dots, y_n\}_{r_1,\dots,r_i,1,\dots,r_k-1,\dots,r_n}$$

$$+ \sum_{k=1}^{n} \pm x\{y_1, \dots, y_k, f_k, \dots, y_n\}_{r_1,\dots,r_k-1,1,\dots,r_n}$$

The second sum can be simplified, depending on the parity of $|y_i|$ for every i. If $|y_i|$ is even, then by formula (iv) of Theorem 1.1.3, we have

$$x\{y_1,\ldots,y_i,y_i,\ldots,y_k,\ldots,y_n\}_{r_1,\ldots,r_i,1,\ldots,r_k-1,\ldots,r_n} = (r_i+1)x\{y_1,\ldots,y_i,\ldots,y_k,\ldots,y_n\}_{r_1,\ldots,r_i+1,\ldots,r_k-1,\ldots,r_n}.$$

If $|y_i|$ is odd, then since we have supposed that any odd degree element has an associated weight equal to 1, we have that $r_i = 1$. By using the symmetry relation, and using that we have no 2-torsion elements since L is free and that $\mathbb{K}$ is an integral domain, we have that

$$x\{y_1,\ldots,y_i,y_i,\ldots,y_k,\ldots,y_n\}_{r_1,\ldots,1,1,\ldots,r_k-1,\ldots,r_n} = 0.$$

We finally have that

$$dl(\mathcal{O}F_{\sum_i r_i}(x,\underbrace{y_1,\ldots,y_1}_{r_1},\ldots,\underbrace{y_n,\ldots,y_n}_{r_n})) = d(x\{y_1,\ldots,y_n\}_{r_1,\ldots,r_n})$$

$$= d(x)\{y_1,\ldots,y_n\}_{r_1,\ldots,r_n} + \sum_{k=1}^n \sum_{\substack{i=1\\ i\neq k}}^n \pm\delta_i\lambda_{k,i}(r_i+1)x\{y_1,\ldots,y_i,\ldots,y_k,\ldots,y_n\}_{r_1,\ldots,r_i+1,\ldots,r_k-1,\ldots,r_n}$$

$$+\sum_{k=1}^n \pm x\{y_1,\ldots,y_k,f_k,\ldots,y_n\}_{r_1,\ldots,r_k-1,1,\ldots,r_n}$$

where $\delta_i = 0$ if $|y_i|$ is odd, and $\delta_i = 1$ else.

We now compute $ld(\mathcal{O}F_{\sum_i r_i}(x,\underbrace{y_1,\ldots,y_1}_{r_1},\ldots,\underbrace{y_n,\ldots,y_n}_{r_n}))$. By the same computations as the beginning of the proof, we have that

$$d\mathcal{O}F_{\sum_i r_i}(x,\underbrace{y_1,\ldots,y_1}_{r_1},\ldots,\underbrace{y_n,\ldots,y_n}_{r_n}) = \mathcal{O}F_{\sum_i r_i}(d(x),\underbrace{y_1,\ldots,y_1}_{r_1},\ldots,\underbrace{y_n,\ldots,y_n}_{r_n})$$

$$+\sum_{k=1}^n \sum_{\widetilde{\sigma}\in Sh(1,r_1,\ldots,r_k-1,1,\ldots,r_n)} \pm\widetilde{\sigma}\cdot(F_{\sum_i r_i}\otimes x\otimes y_1^{\otimes r_1}\otimes\cdots\otimes y_k^{\otimes r_k-1}\otimes d(y_k)\otimes\cdots\otimes y_n^{\otimes r_n}).$$

We now fix $1\leq k\leq n$. We aim to compute the sum

$$\sum_{\widetilde{\sigma}\in Sh(1,r_1,\ldots,r_k-1,1,\ldots,r_n)} \pm\widetilde{\sigma}\cdot(F_{\sum_i r_i}\otimes x\otimes y_1^{\otimes r_1}\otimes\cdots\otimes y_k^{\otimes r_k-1}\otimes d(y_k)\otimes\cdots\otimes y_n^{\otimes r_n}).$$

We use the decomposition of $d(y_k)$ in the chosen basis. The f_k part will precisely give

$$\mathcal{O}F_{\sum_i r_i}(x,\underbrace{y_1,\ldots,y_1}_{r_1},\ldots,\underbrace{y_k,\ldots,y_k}_{r_k-1},f_k,\ldots,\underbrace{y_n,\ldots,y_n}_{r_n}).$$

We now look at the other terms. These terms give the sum

$$\sum_{\substack{i=1\\ i\neq k}}^n \sum_{\widetilde{\sigma}\in Sh(1,r_1,\ldots,r_k-1,1,\ldots,r_n)} \pm\lambda_{k,i}(\widetilde{\sigma}\cdot(F_{\sum_i r_i}\otimes x\otimes y_1^{\otimes r_1}\otimes\cdots\otimes y_k^{\otimes r_k-1}\otimes y_i\otimes\cdots\otimes y_n^{\otimes r_n})).$$

By putting the only y_i with the others, we find that this sum is equal to

$$\sum_{\substack{i=1\\i\neq k}}^{n}\sum_{\widetilde{\sigma}\in Sh(1,r_1,\dots,r_i,1,\dots,r_k-1,\dots,r_n)} \pm\lambda_{k,i}(\widetilde{\sigma}\cdot(F_{\sum_i r_i}\otimes x\otimes y_1^{\otimes r_1}\otimes\cdots\otimes y_i^{\otimes r_i}\otimes y_i\otimes\cdots\otimes y_k^{\otimes r_k-1}\otimes\cdots\otimes y_n^{\otimes r_n})).$$

We now use that for every $\widetilde{\sigma}\in Sh(1,r_1,\dots,r_i,1,\dots,r_k-1,\dots,r_n)$, there exists a unique permutation $\tau_{i,j}$, defined by inserting the last occurrence of y_i among $\underbrace{y_i\otimes\cdots\otimes y_i}_{r_i}$ in position j (so that $1\leq j\leq r_i+1$), such that $\widetilde{\sigma}\tau_{i,j}^{-1}\in Sh(1,r_1,\dots,r_i+1,\dots,r_k-1,\dots,r_n)$. In the other direction, for every $\sigma\in Sh(1,r_1,\dots,r_i+1,\dots,r_k-1,\dots,r_n)$ and $1\leq j\leq r_i+1$, we have that $\sigma\tau_{i,j}\in Sh(1,r_1,\dots,r_i,1,\dots,r_k-1,\dots,r_n)$. We thus obtain the sum

$$\sum_{\substack{i=1\\i\neq k}}^{n}\sum_{\sigma\in Sh(1,r_1,\dots,r_i+1,\dots,r_k-1,\dots,r_n)}\sum_{j=1}^{r_i+1} \pm\lambda_{k,i}(\sigma\tau_{i,j}\cdot(F_{\sum_i r_i}\otimes x\otimes y_1^{\otimes r_1}\otimes\cdots\otimes y_i^{\otimes r_i+1}\otimes\cdots\otimes y_k^{\otimes r_k-1}\otimes\cdots\otimes y_n^{\otimes r_n})).$$

For a fixed $i\neq k$, we need to distinguish two cases: either $|y_i|$ is even, or $|y_i|$ is odd. In the first case, we have that this sum is

$$\sum_{\substack{i=1\\i\neq k}}^{n}\sum_{\sigma\in Sh(1,r_1,\dots,r_i+1,\dots,r_k-1,\dots,r_n)}\sum_{j=1}^{r_i+1} \pm\lambda_{k,i}(\sigma\cdot(F_{\sum_i r_i}\otimes x\otimes y_1^{\otimes r_1}\otimes\cdots\otimes y_i^{\otimes r_i+1}\otimes\cdots\otimes y_k^{\otimes r_k-1}\otimes\cdots\otimes y_n^{\otimes r_n}))$$

which is precisely

$$\sum_{\substack{i=1\\i\neq k}}^{n}\pm\lambda_{k,i}(r_i+1)\mathcal{O}F_{\sum_i r_i}(x,\underbrace{y_1,\dots,y_1}_{r_1},\dots,\underbrace{y_i,\dots,y_i}_{r_i+1},\dots,\underbrace{y_k,\dots,y_k}_{r_k-1},\dots,\underbrace{y_n,\dots,y_n}_{r_n}).$$

In the second case, since we have supposed that every odd degree element has an associated weight equal to 1, we have that $r_i=1$. We have that $\tau_{i,1}$ permutes the two y_i's, which gives a sign, while $\tau_{i,2}=id$. The sum of the two obtained elements is then 0.

We thus have proved that

$$d\mathcal{O}F_{\sum_i r_i}(x,\underbrace{y_1,\dots,y_1}_{r_1},\dots,\underbrace{y_n,\dots,y_n}_{r_n}) = \mathcal{O}F_{\sum_i r_i}(d(x),\underbrace{y_1,\dots,y_1}_{r_1},\dots,\underbrace{y_n,\dots,y_n}_{r_n})$$
$$+\sum_{k=1}^{n}\sum_{\substack{i=1\\ i\neq k}}^{n}\pm\delta_i\lambda_{k,i}(r_i+1)\mathcal{O}F_{\sum_i r_i}(x,\underbrace{y_1,\dots,y_1}_{r_1},\dots,\underbrace{y_i,\dots,y_i}_{r_i+1},\dots,\underbrace{y_k,\dots,y_k}_{r_k-1},\dots,\underbrace{y_n,\dots,y_n}_{r_n})$$
$$+\sum_{k=1}^{n}\pm\mathcal{O}F_{\sum_i r_i}(x,\underbrace{y_1,\dots,y_1}_{r_1},\dots,\underbrace{y_k,\dots,y_k}_{r_k-1},f_k,\dots,\underbrace{y_n,\dots,y_n}_{r_n}).$$

where we have set $\delta_i = 0$ if $|y_i|$ is odd, and $\delta_i = 1$ else. By definition of l, we have

$$l(d\mathcal{O}F_{\sum_i r_i}(x,\underbrace{y_1,\dots,y_1}_{r_1},\dots,\underbrace{y_n,\dots,y_n}_{r_n})) = d(x)\{y_1,\dots,y_n\}_{r_1,\dots,r_n}$$
$$+\sum_{k=1}^{n}\sum_{\substack{i=1\\ i\neq k}}^{n}\pm\delta_i\lambda_{k,i}(r_i+1)x\{y_1,\dots,y_i,\dots,y_k,\dots,y_n\}_{r_1,\dots,r_i+1,\dots,r_i-1,\dots,r_n}$$
$$+\sum_{k=1}^{n}\pm x\{y_1,\dots,y_k,f_k,\dots,y_n\}_{r_1,\dots,r_k-1,1,\dots,r_n}$$

which proves that l commutes with d. $\square$

We then deduce from Propositions 1.2.6 and 1.2.8 that every differential graded pre-Lie algebra with divided powers is in particular a differential graded pre-Lie algebra, with

$$x\star y = x\{y\}_1.$$

Remark 1.2.9. *If $\mathbb{Q}\subset\mathbb{K}$ and if L is a differential graded pre-Lie algebra, then L is a differential graded pre-Lie algebra with divided powers whose weighted braces are explicitly given by*

$$x\{y_1,\dots,y_n\}_{r_1,\dots,r_n} = \frac{1}{\prod_i r_i!}x\{\underbrace{y_1,\dots,y_1}_{r_1},\dots,\underbrace{y_n,\dots,y_n}_{r_n}\}$$

in terms of symmetric braces.

Remark 1.2.10. *Every morphism of $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebras preserves the weighted braces:*

$$f(x\{y_1,\dots,y_n\}_{r_1,\dots,r_n}) = f(x)\{f(y_1),\dots,f(y_n)\}_{r_1,\dots,r_n}.$$

In order to perform infinite sums, we define the notion of a *complete* $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra. We recall the following definition.

Definition 1.2.11. *A* filtered *dg module is the data of a dg module L with inclusions of dg modules*

$$\cdots\subset F_nL\subset F_{n-1}L\subset\cdots\subset F_1L = L.$$

A filtered dg module is complete *if the morphism $L\longrightarrow \lim_{n\geq 1} L/F_nL$ is an isomorphism.*

In general, for every filtered dg module L, the dg module $\widehat{L} = \lim_{n\geq 1} L/F_nL$ is a complete dg module with the filtration $F_n\widehat{L} = Ker(\widehat{L} \longrightarrow L/F_nL)$, since $\widehat{L}/F_n\widehat{L} \simeq L/F_nL$.

We now define the notion of a filtered $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra.

Definition 1.2.12. *A* filtered $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$*-algebra is a* $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$*-algebra endowed with a filtration preserved by the weighted braces in the sense that*

$$F_kL\{F_{k_1}L, \ldots, F_{k_n}L\}_{r_1,\ldots,r_n} \subset F_{k+k_1r_1+\cdots+k_nr_n}L.$$

A filtered $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$*-algebra is* complete *if L is complete as a filtered dg module.*

If L is a filtered $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra, then the weighted braces $-\{-, \ldots, -\}_{r_1,\ldots,r_n}$ induce weighted braces on the completion $\widehat{L} = \lim_{n\geq 1} L/F_nL$, which satisfy the formulas of Theorems 1.2.6 and 1.2.8, and preserve the filtration on $\widehat{L}$, so that $\widehat{L}$ forms a complete $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra (provided that we work over a field).

Examples of $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebras

We give examples of dg pre-Lie algebras with divided powers. The first examples are given by dg brace algebras, following the idea of the proof in the non graded framework in [Ces18].

Definition 1.2.13. *A* differential graded brace algebra *is a differential graded module L endowed with brace operations*

$$-\langle -, \ldots, -\rangle : L^{\otimes n+1} \longrightarrow L$$

which are compatible with the differential d:

$$d(f\langle g_1, \ldots, g_n\rangle) = d(f)\langle g_1, \ldots, g_n\rangle + \sum_{k=1}^{n} \pm f\langle g_1, \ldots, d(g_k), \ldots, g_n\rangle,$$

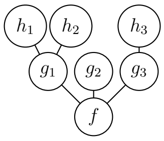
and such that $f\langle\rangle = f$ and

$$f\langle g_1, \ldots, g_n\rangle\langle h_1, \ldots, h_r\rangle = \sum \pm f\langle H_1, g_1\langle H_2\rangle, \ldots, H_{2n-1}, g_n\langle H_{2n}\rangle, H_{2n+1}\rangle,$$

where the sum is over all consecutive subsets $H_1 \sqcup H_2 \sqcup \cdots \sqcup H_{2n+1} = \{h_1, \ldots, h_r\}$, and the sign is yielded by the permutation of the g_i's with the h_j's.

The operad which governs brace algebras is denoted by $\mathcal{B}race$, and is defined, in arity n, as the $\mathbb{K}$-module spanned by the planar n-trees, i.e. trees with an order on the set of inputs for each vertex (see [Ces18, §6.1] or [Cha02, §2] for some details on the operad $\mathcal{B}race$).

This operad allows us to represent all operations in brace algebras by the action of a planar tree, or by a planar tree labeled with the inputs. For instance, we have



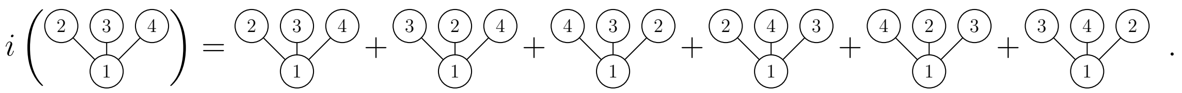
$$= f\langle g_1\langle h_1,h_2\rangle, g_2, g_3\langle h_3\rangle\rangle.$$

Remark 1.2.14. *Because the action of the symmetric groups on $\mathcal{B}race$ is free, we have that the trace map induces an isomorphism of monads $Tr : \mathcal{S}(\mathcal{B}race,-) \longrightarrow \Gamma(\mathcal{B}race,-)$.*

We have an inclusion

$$i : \mathcal{P}re\mathcal{L}ie \hookrightarrow \mathcal{B}race$$

defined by the *symmetrization* of trees. Namely, i is obtained by summing over all possible ways to write a given tree t as a planar tree. For instance:



The map i induces a morphism of monads that can be used to define a $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra structure on every dg brace algebra L, given by the following composition:

$$\Gamma(\mathcal{P}re\mathcal{L}ie,L) \xrightarrow{\Gamma(i,L)} \Gamma(\mathcal{B}race,L) \xleftarrow[\simeq]{Tr} \mathcal{S}(\mathcal{B}race,L) \xrightarrow{\quad l \quad} L,$$

where we denote by $l : \mathcal{S}(\mathcal{P}re\mathcal{L}ie,L) \longrightarrow L$ the $\mathcal{B}race$-algebra structure. We aim to compute the weighted braces.

Theorem 1.2.15. *Every dg brace algebra L is endowed with a $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra structure. Moreover, weighted braces $-\{-,\dots,-\}_{r_1,\dots,r_n}$ are explicitly given by*

$$f\{g_1,\dots,g_n\}_{r_1,\dots,r_n} = \sum_{\sigma\in Sh(r_1,\dots,r_n)} \pm f\langle \overline{g}_{\sigma^{-1}(1)},\dots,\overline{g}_{\sigma^{-1}(r)}\rangle,$$

where we have set $r=\sum_i r_i$ and $(\overline{g}_1,\dots,\overline{g}_r) = (\underbrace{g_1,\dots,g_1}_{r_1},\dots,\underbrace{g_n,\dots,g_n}_{r_n})$.

Proof. Let L be a brace algebra. As seen before, we have a $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra structure on L given by the composite

$$\Gamma(\mathcal{P}re\mathcal{L}ie,L) \xrightarrow{\Gamma(i,L)} \Gamma(\mathcal{B}race,L) \xleftarrow[\simeq]{Tr} \mathcal{S}(\mathcal{B}race,L) \xrightarrow{\quad l \quad} L$$

where $l : \mathcal{S}(\mathcal{B}race,L) \longrightarrow L$ is the brace algebra structure. We now compute the weighted braces. Let $f,g_1,\dots,g_n \in L$ be homogeneous elements with $g_i \neq g_j$ whenever $i\neq j$ and $r_1,\dots,r_n \geq 0$ (recall that we have suppose that, in the situation $char(\mathbb{K})\neq 2$, any odd degree element has an associated weight equaled to 0 or 1). We set $E = E_{f,g_1,\dots,g_n}$ and $\psi = \psi_{f,g_1,\dots,g_n}$ (see the proof of Theorem 1.2.8). We use the following commutative diagram

$$\begin{tikzcd}
\Gamma(\mathcal{P}re\mathcal{L}ie,L) \arrow[r, "\Gamma(i,L)"] & \Gamma(\mathcal{B}race,L) & \mathcal{S}(\mathcal{B}race,L) \arrow[l, "Tr"', "\simeq"] \arrow[r, "l"] & L \\
\Gamma(\mathcal{P}re\mathcal{L}ie,E) \arrow[u, "\Gamma(\mathcal{P}re\mathcal{L}ie,\psi)"] \arrow[r, "\Gamma(i,E)"'] & \Gamma(\mathcal{B}race,E) \arrow[u, "\Gamma(\mathcal{B}race,\psi)"] & \mathcal{S}(\mathcal{B}race,E). \arrow[l, "\simeq", "Tr"'] \arrow[u, "\mathcal{S}(\mathcal{B}race,\psi)"]
\end{tikzcd}$$

We keep the notations $f, g_1, \ldots, g_n$ for the corresponding elements in E. Then the element $f\{g_1, \ldots, g_n\}_{r_1,\ldots,r_n}$ is given by the image of $x = \mathcal{O}F_r(f, \underbrace{g_1, \ldots, g_1}_{r_1}, \ldots, \underbrace{g_n, \ldots, g_n}_{r_n}) \in \Gamma(\mathcal{P}re\mathcal{L}ie, E)$ under the composite

$$\Gamma(\mathcal{P}re\mathcal{L}ie, E) \xrightarrow{\Gamma(\mathcal{P}re\mathcal{L}ie,\psi)} \Gamma(\mathcal{P}re\mathcal{L}ie, L) \xrightarrow{\Gamma(i,L)} \Gamma(\mathcal{B}race, L) \xleftarrow[\simeq]{Tr} \mathcal{S}(\mathcal{B}race, L) \xrightarrow{l} L.$$

Our goal is to compute the image of x under the bottom composite of the diagram, which is

$$\Gamma(\mathcal{P}re\mathcal{L}ie, E) \xrightarrow{\Gamma(i,E)} \Gamma(\mathcal{B}race, E) \xleftarrow[\simeq]{Tr} \mathcal{S}(\mathcal{B}race, E) \xrightarrow{\mathcal{S}(\mathcal{B}race,\psi)} \mathcal{S}(\mathcal{B}race, L).$$

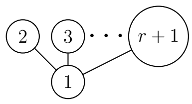
We set $(\overline{g}_1, \ldots, \overline{g}_{r+1}) = (f, \underbrace{g_1, \ldots, g_1}_{r_1}, \ldots, \underbrace{g_n, \ldots, g_n}_{r_n})$ (note that we have added f here so that these $\overline{g}_i$'s are different from the $\overline{g}_i$'s of the theorem). We then precisely have:

$$x = \sum_{\sigma \in \Sigma_{r+1}/\prod_i \Sigma_{r_i}} \pm(\sigma \cdot F_r) \otimes \overline{g}_{\sigma^{-1}(1)} \otimes \cdots \otimes \overline{g}_{\sigma^{-1}(r+1)}$$

by definition of the orbit map (see Lemma 1.2.4). Now, because $\Sigma_{r+1}/\prod_i \Sigma_{r_i}$ is in bijection with $Sh(1, r_1, \ldots, r_n)$, we can write x as

$$x = \sum_{\sigma \in Sh(1,r_1,\ldots,r_n)} \pm(\sigma \cdot F_r) \otimes \overline{g}_{\sigma^{-1}(1)} \otimes \cdots \otimes \overline{g}_{\sigma^{-1}(r+1)}.$$

We now embed $\mathcal{P}re\mathcal{L}ie$ into $\mathcal{B}race$. The tree F_r can be seen in $\mathcal{B}race$ as $\sum_{\substack{s \in \Sigma_{r+1} \\ s(1)=1}} s \cdot \overline{F_r}$ where $\overline{F_r}$ is the planar tree

$\overline{F_r} =$ 

We then obtain, in $\Gamma(\mathcal{B}race, E)$,

$$x = \sum_{\sigma \in Sh(1,r_1,\ldots,r_n)} \sum_{\substack{s \in \Sigma_{r+1} \\ s(1)=1}} \pm(\sigma s \cdot \overline{F_r}) \otimes \overline{g}_{\sigma^{-1}(1)} \otimes \cdots \otimes \overline{g}_{\sigma^{-1}(r+1)}.$$

We now need to compute $y = Tr^{-1}(x) \in \mathcal{S}(\mathcal{B}race, E)$. We claim that

$$y = \sum_{\substack{\omega \in Sh(1,r_1,\ldots,r_n) \\ \omega(1)=1}} \pm \overline{F_r}(\overline{g}_{\omega^{-1}(1)}, \ldots, \overline{g}_{\omega^{-1}(r+1)}).$$

We compute

$$Tr(y) = \sum_{\substack{\omega \in Sh(1,r_1,\ldots,r_n) \\ \omega(1)=1}} \sum_{\tau \in \Sigma_{r+1}} \pm(\tau \cdot \overline{F_r}) \otimes \overline{g}_{\omega^{-1}\tau^{-1}(1)} \otimes \cdots \otimes \overline{g}_{\omega^{-1}\tau^{-1}(r+1)}.$$

The fact that $Tr(y) = x$ comes from the existence of a bijective correspondance

$$\varphi : Sh(1, r_1, \ldots, r_n)\times\{s \in \Sigma_{r+1} \mid s(1) = 1\} \longrightarrow \{\omega \in Sh(1, r_1, \ldots, r_n) \mid \omega(1) = 1\}\times\Sigma_{r+1}.$$

Indeed, let $\sigma \in Sh(1, r_1, \ldots, r_n)$ and $s \in \Sigma_{r+1}$ be such that $s(1) = 1$. We set $\tau = \sigma s$, and decompose $\tau^{-1}\sigma = s^{-1} = \omega\mu$ as a product of $\omega \in Sh(1, r_1, \ldots, r_n)$ with $\mu \in \Sigma_1 \times \Sigma_{r_1} \times \cdots \times \Sigma_{r_n}$. Since $s(1) = 1$, we have that $\omega(1) = 1$. We then set $\varphi(\sigma, s) := (\omega, \tau)$. Since the couple (ω, μ) uniquely depends on s, we have a well defined injective map φ between two sets with the same cardinal. The map φ is then a bijection.

Since we have $\overline{g}_{\mu\sigma^{-1}(i)} = \overline{g}_{\sigma^{-1}(i)}$ for every i, we obtain that $Tr(y) = x$, which proves the theorem. □

Corollary 1.2.16. *Let $\mathcal{P}$ be a non symmetric dg operad with $\mathcal{P}(0) = 0$. We denote by $1 \in \mathcal{P}(1)$ the unit element of $\mathcal{P}$, by $p(q_1, \ldots, q_n) = p \otimes q_1 \otimes \cdots \otimes q_n \in \mathcal{P} \circ \mathcal{P}$ and by $\gamma : \mathcal{P} \circ \mathcal{P} \longrightarrow \mathcal{P}$ the operadic composition of $\mathcal{P}$. Then the dg module $\bigoplus_{r\geq 1} \mathcal{P}(r)$ admits a structure of a $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra induced by the following brace algebra structure:*

$$p\langle q_1, \ldots, q_n\rangle = \sum_{1\leq i_1<\cdots<i_n\leq r} \gamma(p(1, \ldots, \underset{i_1}{q_1}, \ldots, \underset{i_n}{q_n}, \ldots, 1)).$$

We also set $p\langle q_1, \ldots, q_n\rangle = 0$ if the operadic composition is not possible.

Proof. We refer to [GV95] for the brace algebra structure of $\bigoplus_{r\geq 1} \mathcal{P}(r)$. It is endowed with a $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra structure by Theorem 1.2.15. □

In the symmetric context, we can recover an analogue of this corollary for $\bigoplus_{r\geq 1} \mathcal{P}(r)^{\Sigma_r}$. However, the operations $-\langle -, \ldots, -\rangle$ do not preserve $\bigoplus_{r\geq 1} \mathcal{P}(r)^{\Sigma_r}$. We thus need to force the symmetry, and then to sum on every possible positions of $q_1, \ldots, q_n$ in order to retrieve a $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra structure.

Proposition 1.2.17. *Suppose that $\mathbb{K}$ is a field and let $\mathcal{P}$ be a symmetric dg operad such that $\mathcal{P}(0) = 0$. Then $\mathcal{L}(\mathcal{P}) = \bigoplus_{r\geq 1} \mathcal{P}(r)^{\Sigma_r}$ is endowed with a $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra structure defined by*

$$p\{q_1, \ldots, q_n\}_{r_1,\ldots,r_n} = \sum_{\sigma\in Sh(r_1,\ldots,r_n)} \sum_{\substack{1\leq i_1<\cdots<i_r\leq u \\ \omega\in Sh_*(1,\ldots,\underset{i_1}{\overline{s}_{\sigma^{-1}(1)}},\ldots,\underset{i_r}{\overline{s}_{\sigma^{-1}(r)}},\ldots,1)}} \pm\omega\cdot\gamma(p(1, \ldots, \underset{i_1}{\overline{q}_{\sigma^{-1}(1)}}, \ldots, \underset{i_r}{\overline{q}_{\sigma^{-1}(r)}}, \ldots, 1)),$$

for elements of homogeneous arity $p \in \mathcal{P}(m)^{\Sigma_m}, q_1 \in \mathcal{P}(s_1)^{\Sigma_{s_1}}, \ldots, q_n \in \mathcal{P}(s_n)^{\Sigma_{s_n}}$ and where we have set $r = \sum_i r_i$, $(\overline{q}_1, \ldots, \overline{q}_r) = (\underbrace{q_1, \ldots, q_1}_{r_1}, \ldots, \underbrace{q_n, \ldots, q_n}_{r_n})$ and $(\overline{s}_1, \ldots, \overline{s}_r) = (\underbrace{s_1, \ldots, s_1}_{r_1}, \ldots, \underbrace{s_n, \ldots, s_n}_{r_n})$. The sign is induced by the commutation of $\overline{q}_1, \ldots, \overline{q}_r$ to $\overline{q}_{\sigma^{-1}(r)}, \ldots, \overline{q}_{\sigma^{-1}(1)}$. We also set $p\{q_1, \ldots, q_n\}_{r_1,\ldots,r_n} = 0$ if $u < r_1 + \cdots + r_n$. The weighted brace operations are then extended to the sum $\bigoplus_{r\geq 1} \mathcal{P}(r)^{\Sigma_r}$ by using Formula (v) of Theorem 1.1.3.

Proof. We first prove that these operations preserve $\mathcal{L}(\mathcal{P})$. Let $p \in \mathcal{P}(m)^{\Sigma_m}, q_1 \in \mathcal{P}(s_1)^{\Sigma_{s_1}}, \dots, q_n \in \mathcal{P}(s_n)^{\Sigma_{s_n}}$ and $r_1, \dots, r_n \geq 0$. Notice that since we have

$$p\{q_1, \dots, q_n\}_{r_1,\dots,r_n} = p\{q_1, \dots, q_n, 1\}_{r_1,\dots,r_n,m-(r_1+\dots+r_n)},$$

we can suppose that $m = r_1 + \dots + r_n$. We then have

$$p\{q_1, \dots, q_n\}_{r_1,\dots,r_n} = \sum_{\sigma \in Sh(r_1,\dots,r_n)} \sum_{\omega \in Sh_*(\overline{s}_{\sigma^{-1}(1)},\dots,\overline{s}_{\sigma^{-1}(r)})} \pm\omega \cdot \gamma(p(\overline{q}_{\sigma^{-1}(1)}, \dots, \overline{q}_{\sigma^{-1}(r)})).$$

Let $\mu \in \Sigma_{s_1 r_1 + \dots + s_n r_n}$. For a given $\sigma \in Sh(r_1, \dots, r_n)$, we write $\mu\omega = \widetilde{\omega} \cdot \nu(\tau_1, \dots, \tau_r)$ where $\nu \in \Sigma_r, \tau_1 \in \Sigma_{\overline{s}_{\sigma^{-1}(1)}}, \dots, \tau_r \in \Sigma_{\overline{s}_{\sigma^{-1}(r)}}, \widetilde{\omega} \in Sh_*(\overline{s}_{(\nu\sigma)^{-1}(1)}, \dots, \overline{s}_{(\nu\sigma)^{-1}(r)})$. We also have set $\nu(\tau_1, \dots, \tau_r)$ to be the composite of $\tau_1 \oplus \dots \oplus \tau_r$ with the corresponding blocks permutation given by $\nu \in \Sigma_r$. We obtain

$$\mu\omega \cdot \gamma(p(\overline{q}_{\sigma^{-1}(1)}, \dots, \overline{q}_{\sigma^{-1}(r)})) = \widetilde{\omega} \cdot \gamma(p(\overline{q}_{(\nu\sigma)^{-1}(1)}, \dots, \overline{q}_{(\nu\sigma)^{-1}(r)})),$$

as $p, q_1, \dots, q_n$ are invariants. We now write $\nu\sigma = \widetilde{\sigma} \cdot (\widetilde{\tau_1} \oplus \dots \oplus \widetilde{\tau_r})$ where $\widetilde{\sigma} \in Sh(r_1, \dots, r_n), \widetilde{\tau_1} \in \Sigma_{r_1}, \dots, \widetilde{\tau_r} \in \Sigma_{r_n}$. We thus obtain

$$\mu\omega \cdot \gamma(p(\overline{q}_{\sigma^{-1}(1)}, \dots, \overline{q}_{\sigma^{-1}(r)})) = \widetilde{\omega} \cdot \gamma(p(\overline{q}_{\widetilde{\sigma}^{-1}(1)}, \dots, \overline{q}_{\widetilde{\sigma}^{-1}(r)})).$$

We thus have proved that

$$\begin{aligned} \mu \cdot (p\{q_1, \dots, q_n\}_{r_1,\dots,r_n}) = \\ \sum_{\widetilde{\sigma} \in Sh(r_1,\dots,r_n)} \sum_{\widetilde{\omega} \in Sh_*(\overline{s}_{\widetilde{\sigma}^{-1}(1)},\dots,\overline{s}_{\widetilde{\sigma}^{-1}(r)})} \pm\widetilde{\omega} \cdot \gamma(p(\overline{q}_{\widetilde{\sigma}^{-1}(1)}, \dots, \overline{q}_{\widetilde{\sigma}^{-1}(r)})) \\ = p\{q_1, \dots, q_n\}_{r_1,\dots,r_n}. \end{aligned}$$

The operations $-\{-, \dots, -\}_{r_1,\dots,r_n}$ then preserve $\mathcal{L}(\mathcal{P})$.

We now prove formulas of Theorem 1.2.6. We can immediately check that formulas $(i) - (v)$ are satisfied. The commutation with the differential is also satisfied since the operadic structure is compatible with the differential. It remains to prove formula (vi) of Theorem 1.2.6. We first note that the theorem holds if $\mathbb{K} = \mathbb{Q}$. Indeed, in that case, the trace map $Tr : \mathcal{S}(\mathcal{P}re\mathcal{L}ie, -) \longrightarrow \Gamma(\mathcal{P}re\mathcal{L}ie, -)$ induces an isomorphism of monads. We thus only need to prove that the $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra structure is induced by a pre-Lie algebra structure. This presumed pre-Lie algebra structure is given by

$$p\{q\}_1 = \sum_{i=1}^{m} \sum_{\omega \in Sh_*(1,\dots,\underset{i}{n},\dots,1)} \omega \cdot (p \circ_i q),$$

where $p \in \mathcal{P}(m)^{\Sigma_m}$ and $q \in \mathcal{P}(n)^{\Sigma_n}$. We then recover the pre-Lie algebra structure given in [LV12, §5.3.16]. We now need to prove that the operations $-\{-, \dots, -\}_{1,\dots,1}$ coincide with the symmetric braces $-\{-, \dots, -\}$ induced by the pre-Lie operation (see

Definition 1.1.1). It is equivalent to prove the identity

$$p\{q_1,\ldots,q_{n+1}\}_{1,\ldots,1} = p\{q_1,\ldots,q_n\}_{1,\ldots,1}\{q_{n+1}\}_1 - \sum_{k=1}^{n} \pm p\{q_1,\ldots,q_k\{q_{n+1}\}_1,\ldots,q_n\}_{1,\ldots,1}.$$

This follows from the associativity of the operadic composition. More precisely, the term $p\{q_1,\ldots,q_n\}_{1,\ldots,1}\{q_{n+1}\}_1$ is composed of two types of operadic composition. Either q_{n+1} is in the same level as $q_1,\ldots,q_n$, which will give exactly $p\{q_1,\ldots,q_{n+1}\}_{1,\ldots,1}$ by definition, or q_{n+1} will be attached to one of $q_1,\ldots,q_n$. These last terms are removed in order to retrieve $p\{q_1,\ldots,q_{n+1}\}_{1,\ldots,1}$.

We now prove the general case. Consider elements $p,q_1,\ldots,q_n,f_1,\ldots,f_m$ in $\bigoplus_{r\geq 1}\mathcal{P}(r)^{\Sigma_r}$ which are homogeneous in degrees and in arities, and $r_1,\ldots,r_n,s_1,\ldots,s_m\geq 0$. We need to compute $p\{q_1,\ldots,q_n\}_{r_1,\ldots,r_n}\{f_1,\ldots,f_m\}_{s_1,\ldots,s_m}$ and to find the right hand-side of Theorem 1.2.6. We consider the symmetric sequence $M_{\mathbb{K}}$, defined over any ring $\mathbb{K}$, spanned by abstract variables $P,Q_1,\ldots,Q_n,F_1,\ldots,F_m$ of the same arities and degrees as $p,q_1,\ldots,q_n,f_1,\ldots,f_m$ and endowed with a trivial action of the symmetric groups. We have an obvious morphism of symmetric sequences $M_{\mathbb{K}}\longrightarrow\mathcal{P}$ which sends P to p, the Q_i's to the q_i's and the F_j's to the f_j's. We thus have a unique morphism of operads $\mathcal{F}(M_{\mathbb{K}})\longrightarrow\mathcal{P}$ which extends the morphism $M_{\mathbb{K}}\longrightarrow\mathcal{P}$, where $\mathcal{F}(M_{\mathbb{K}})$ is the free operad generated by the symmetric sequence $M_{\mathbb{K}}$. Because the presumed $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra structure is written in terms of the operadic composition, if formula (vi) of Theorem 1.2.6 holds for $\mathcal{F}(M_{\mathbb{K}})$, then it holds also for $\mathcal{P}$.

We prove first that the formula is satisfied for $\mathcal{F}(M_{\mathbb{Z}})$. Since the morphism of rings $\mathbb{Z}\hookrightarrow\mathbb{Q}$ induces an injective morphism of operads $\mathcal{F}(M_{\mathbb{Z}})\hookrightarrow\mathcal{F}(M_{\mathbb{Q}})$ and since the relation (vi) of Theorem 1.2.6 is true in $\mathcal{F}(M_{\mathbb{Q}})$, then it is also satisfied in $\mathcal{F}(M_{\mathbb{Z}})$. By using the morphism $\mathbb{Z}\longrightarrow\mathbb{K}$ which gives rise to a morphism of operads $\mathcal{F}(M_{\mathbb{Z}})\longrightarrow\mathcal{F}(M_{\mathbb{K}})$, we find that it is also satisfied in $\mathcal{F}(M_{\mathbb{K}})$.

We thus have weighted braces operation on $\bigoplus_{r\geq 1}\mathcal{P}(r)^{\Sigma_r}$. Since we work on a field $\mathbb{K}$, by Theorem 1.2.8, it implies that this structure endows $\bigoplus_{r\geq 1}\mathcal{P}(r)^{\Sigma_r}$ with a structure of a $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra. □

Corollary 1.2.18. *Suppose that $\mathbb{K}$ is a field and let $\mathcal{P}$ be a dg operad such that $\mathcal{P}(0)=0$ and $\mathcal{P}(1)=\mathbb{K}$. Then $\prod_{r\geq 2}\mathcal{P}(r)^{\Sigma_r}$ is a complete $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra.*

Proof. Let $\mathcal{L}(\mathcal{P})=\bigoplus_{r\geq 2}\mathcal{P}(r)^{\Sigma_r}$. Note that $\mathcal{L}(\mathcal{P})$ is a sub $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra of $\bigoplus_{r\geq 1}\mathcal{P}(r)^{\Sigma_r}$. We have a filtration on it given by $F_k\mathcal{L}(\mathcal{P})=\bigoplus_{r\geq k+1}\mathcal{P}(r)^{\Sigma_r}$ which is preserved by the weighted brace operations. The completion with respect to this filtration is exactly $\prod_{r\geq 2}\mathcal{P}(r)^{\Sigma_r}$. By the remark after Definition 1.2.12, we have weighted brace operations on $\prod_{r\geq 2}\mathcal{P}(r)^{\Sigma_r}$. Since we work over a field, this endows $\prod_{r\geq 2}\mathcal{P}(r)^{\Sigma_r}$ with a structure of a $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra by Theorem 1.2.8. □

1.2.2 The gauge group

We can now define an analogue of the circular product given in [DSV16] using the weighted brace operations. Before doing so, we adopt the following notations. Let L

be a complete $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra and $L_+ = \mathbb{K}1 \oplus L$. We extend the weighted braces $-\{-, \dots, -\}_{r_1,\dots,r_n} : L^{\times n+1} \longrightarrow L$ on $L_+ \times L^{\times n}$ by setting

$$1\{y_1, \dots, y_n\}_{r_1,\dots,r_n} = \begin{cases} y_i & \text{if } r_i = 1 \text{ and } \forall k \neq i, r_k = 0 \\ 0 & \text{if } r_1 + \dots + r_n > 1 \end{cases} .$$

for every $y_1, \dots, y_n \in L$. We can check that all the formulas from Theorem 1.1.3 are still satisfied if we take $x \in L_+$.

Definition 1.2.19. *Let $\alpha \in L_+$ and $\mu \in L^0$. We set*

$$\alpha \circledcirc (1 + \mu) = \sum_{n=0}^{+\infty} \alpha\{\mu\}_n.$$

Note that this quantity is well defined since L is complete, and because $1\{y\}_n = 0$ as soon as $n \geq 2$.

By applying this definition in the case $\mathbb{Q} \subset \mathbb{K}$ and using the weighted braces given by Remark 1.2.9, we retrieve the usual circular product given in [DSV16].

Remark 1.2.20. *One can see that we have $1 \circledcirc (1 + \mu) = 1 + \mu = (1 + \mu) \circledcirc 1$ so that 1 is a unit element for $\circledcirc$. We thus have*

$$\forall \mu, \nu \in L^0, (1 + \mu) \circledcirc (1 + \nu) = 1 + \nu + \sum_{n=0}^{+\infty} \mu\{\nu\}_n,$$

which shows that $\circledcirc$ preserves $1 + L^0$.

Lemma 1.2.21. *The circular product $\circledcirc$ is associative, in the sense that for all $\alpha \in L_+$ and $\mu, \nu \in L^0$,*

$$(\alpha \circledcirc (1 + \mu)) \circledcirc (1 + \nu) = \alpha \circledcirc ((1 + \mu) \circledcirc (1 + \nu)).$$

Proof. Let $\alpha \in L_+$ and $\mu, \nu \in L^0$. We first have

$$\begin{aligned} (\alpha \circledcirc (1 + \mu)) \circledcirc (1 + \nu) &= \left(\sum_{n=0}^{+\infty} \alpha\{\mu\}_n \right) \circledcirc (1 + \nu) \\ &= \sum_{n,p=0}^{+\infty} \alpha\{\mu\}_n\{\nu\}_p. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \alpha \circledcirc ((1 + \mu) \circledcirc (1 + \nu)) &= \alpha \circledcirc \left(1 + \nu + \sum_{n=0}^{+\infty} \mu\{\nu\}_n \right) \\ &= \sum_{p=0}^{+\infty} \alpha \left\{ \nu + \sum_{n=0}^{+\infty} \mu\{\nu\}_n \right\}_p . \end{aligned}$$

We thus need to prove that

$$\sum_{n,p=0}^{+\infty} \alpha\{\mu\}_n\{\nu\}_p = \sum_{p=0}^{+\infty} \alpha\left\{\nu + \sum_{n=0}^{+\infty} \mu\{\nu\}_n\right\}_p .$$

To prove this identity, we use formula (vi) of Theorem 1.1.3:

$$\alpha\{\mu\}_n\{\nu\}_p = \sum_{p=\beta+\sum_{i=1}^n \alpha^i} \frac{1}{n!}\alpha\{\mu\{\nu\}_{\alpha^1}, \dots, \mu\{\nu\}_{\alpha^n}, \nu\}_{1,\dots,1,\beta},$$

which gives

$$\sum_{n,p=0}^{+\infty} \alpha\{\mu\}_n\{\nu\}_p = \sum_{n=0}^{+\infty}\sum_{p=0}^{+\infty}\sum_{\beta=0}^{p}\sum_{p-\beta=\sum_{i=1}^n \alpha^i} \frac{1}{n!}\alpha\{\mu\{\nu\}_{\alpha^1}, \dots, \mu\{\nu\}_{\alpha^n}, \nu\}_{1,\dots,1,\beta}.$$

In this sum, because of the symmetry, some terms occur several times. For a given p and β, we count the number of partitions of $p-\beta = \alpha^1 + \cdots + \alpha^n$ of the particular form $r_1\widetilde{\alpha^1} + \cdots + r_q\widetilde{\alpha^q}$. We get $n(\widetilde{\alpha^1}, \dots, \widetilde{\alpha^n}) = \frac{n!}{r_1!\cdots r_q!}$ for this number. We then have

$$\frac{1}{r_1!\cdots r_q!}\alpha\{\mu\{\nu\}_{\widetilde{\alpha}^1}, \dots, \mu\{\nu\}_{\widetilde{\alpha}^1}, \dots, \mu\{\nu\}_{\widetilde{\alpha}^q}, \dots, \mu\{\nu\}_{\widetilde{\alpha}^q}, \nu\}_{1,\dots,1,\beta} =$$
$$\alpha\{\mu\{\nu\}_{\widetilde{\alpha}^1}, \dots, \mu\{\nu\}_{\widetilde{\alpha}^q}, \nu\}_{r_1,\dots,r_q,\beta}.$$

We conclude by formula (v) of Theorem 1.1.3. $\square$

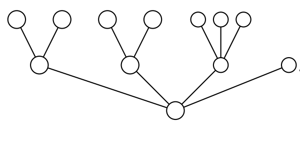
We now need to find an explicit inverse for a given element $1-\mu$ with $\mu \in L^0$.

Definition 1.2.22. *Let t be a non-labeled tree with n vertices and $\mu \in L^0$. We set*

$$\mathcal{O}t(\mu) = \gamma(\mathcal{O}t(\mu^{\otimes n})),$$

for some choice of labeling of t.

Note that because $\mathcal{O}$ is Σ-invariant, this quantity does not depend on the choice of a labeling for t. For example, let t be the non-labeled tree

$t =$ .

Then

$$\mathcal{O}t(\mu) = \mu\{\mu\{\mu\}_2, \mu\{\mu\}_3, \mu\}_{2,1,1}.$$

Lemma 1.2.23. *For every $\mu \in L^0$, the element $1-\mu$ has an inverse in $1+L^0$ for the circular product $\circledcirc$ given by*

$$(1-\mu)^{\circledcirc -1} = 1 + \sum_{t\in rRT^*} \mathcal{O}t(\mu),$$

where rRT^* is the set of trees without any labeling and with at least one vertex.

Proof. We first see that this defines a right-inverse for $1-\mu$. Indeed, we first have that

$$(1-\mu)\circledcirc\left(1+\sum_{t\in rRT^*}\mathcal{O}t(\mu)\right)=1+\sum_{t\in rRT^*}\mathcal{O}t(\mu)-\sum_{k=0}^{+\infty}\mu\left\{\sum_{t\in rRT^*}\mathcal{O}t(\mu)\right\}_k.$$

Then, as every $t\in rRT^*$ can be uniquely described by its root and branches, we have that every term in the first sum at the right hand side can be uniquely described by an element from the second sum, and vice versa. Formula (v) from Theorem 1.1.3 thus give the result.

We now need to prove that it is a left inverse, which is slightly more difficult. We compute

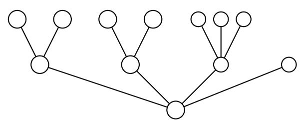
$$\left(1+\sum_{t\in rRT^*}\mathcal{O}t(\mu)\right)\circledcirc(1-\mu)=1-\mu+\sum_{t\in rRT^*}\mathcal{O}t(\mu)+\sum_{k\geq 1}\left(\sum_{t\in rRT^*}\mathcal{O}t(\mu)\right)\{-\mu\}_k.$$

We focus on one term $\mathcal{O}t(\mu)$ from the first sum, for some tree $t\in rRT^*$. Recall that a vertex of t is called a *leaf* if it is not the root of t and if it is connected to one and only one other vertex in t. We denote by m_t the number of leaves.

If $m_t=0$, then t is the trivial tree: $\mathcal{O}t(\mu)=\mu$. This term does not appear in the second sum (because $k\geq 1$) and vanishes with $-\mu$.

If $m_t\neq 0$, the idea is to fix a number $1\leq k\leq m_t$, and to see which trees we can obtain if we remove k leaves of t. These trees will occur in the second sum and give $(-1)^k\mathcal{O}t(\mu)$ by adding k copies of $-\mu$.

Let X_t be the set of leaves of t. Let $X_{t,k}$ be the set of non ordered subsets of X_t with k elements. When we remove k leaves, we need to take into account that we can obtain the same tree by removing a different non ordered set of k leaves. For example, if we take the previous tree

$t=$ 

and if we look at the first branch, removing the vertex at the left gives the same tree as removing the vertex at the right.

Let $t_k^1,\dots,t_k^{p_k}$ be all the different trees that we can get from t by removing k leaves. We denote by $X_{t,k}^{t_k^i}$ the subset of $X_{t,k}$ formed by all the vertices that lead to t_k^i when removing them from t. We then have a disjoint union $X_{t,k}=\displaystyle\bigsqcup_{i=1}^{p_k}X_{t,k}^{t_k^i}$.

Each terms $\mathcal{O}t_k^i(\mu)\{-\mu\}_k$ will then give, among other terms, $(-1)^kCard(X_{t,k}^{t_k^i})\mathcal{O}t(\mu)$. When we take the sum over i, we obtain $(-1)^kCard(X_{t,k})\mathcal{O}t(\mu)=(-1)^k\binom{m_t}{k}\mathcal{O}t(\mu)$.

By taking the sum over k, we therefore obtain $-\mathcal{O}t(\mu)$ which vanishes with $\mathcal{O}t(\mu)$ given by the first sum. $\square$

From Lemma 1.2.21 and Lemma 1.2.23, we deduce assertion (i) of Theorem A:

Theorem 1.2.24. *The triple $G = (1 + L^0, \circledcirc, 1)$ is a group called the gauge group of L.*

1.2.3 Maurer-Cartan elements and the Deligne groupoid

We now aim to prove assertion (ii) of Theorem A. We first make explicit the definition of the Maurer-Cartan set.

Definition 1.2.25. *Let L be a dg $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra. A given $\alpha \in L^1$ is a* Maurer-Cartan element *if it satisfies the* Maurer-Cartan equation*:*

$$d(\alpha) + \alpha\{\alpha\}_1 = 0.$$

We let $\mathcal{MC}(L)$ to be the set of all Maurer-Cartan elements of L.

Remark 1.2.26. *In the case $\mathbb{Q} \subset \mathbb{K}$, we retrieve the classical definition:*

$$d(\alpha) + \frac{1}{2}[\alpha, \alpha] = 0,$$

written with the dg Lie algebra structure on L.

As in the case of characteristic zero, we expect the gauge group to act on the Maurer-Cartan set. Before seeing that, we define a new operation.

Definition 1.2.27. *Let $\alpha \in L_+, \beta \in L$ and $1 + \mu \in G$. We set*

$$\alpha \circledcirc (1 + \mu; \beta) = \sum_{n=0}^{+\infty} \alpha\{\mu, \beta\}_{n,1}.$$

Lemma 1.2.28. *We have the following identities:*

$$\begin{array}{lcl} (\alpha \circledcirc (1+\mu))\{\beta\}_1 & = & \alpha \circledcirc (1+\mu; \beta + \mu\{\beta\}_1), \\ \alpha\{\beta\}_1 \circledcirc (1+\mu) & = & \alpha \circledcirc (1+\mu; \beta \circledcirc (1+\mu)), \\ d(\alpha \circledcirc (1+\mu)) & = & d(\alpha) \circledcirc (1+\mu) + (-1)^{|\alpha|}\alpha \circledcirc (1+\mu; d(\mu)). \end{array}$$

Proof. By applying formula (vi) of Theorem 1.1.3, we find that

$$\begin{aligned} (\alpha \circledcirc (1+\mu))\{\beta\}_1 &= \sum_{n=0}^{+\infty} \alpha\{\mu\}_n\{\beta\}_1 \\ &= \sum_{n=0}^{+\infty} \alpha\{\mu, \beta\}_{n,1} + \sum_{n=1}^{+\infty} \alpha\{\mu, \mu\{\beta\}_1\}_{n-1,1} \\ &= \sum_{n=0}^{+\infty} \alpha\{\mu, \beta + \mu\{\beta\}_1\}_{n,1} \\ &= \alpha \circledcirc (1+\mu; \beta + \mu\{\beta\}_1), \end{aligned}$$

as well as

$$\begin{aligned}\alpha\{\beta\}_1 \circledcirc (1+\mu) &= \sum_{m=0}^{+\infty} \alpha\{\beta\}_1\{\mu\}_m \\ &= \sum_{p,q=0}^{+\infty} \alpha\{\beta\{\mu\}_p, \mu\}_{1,q} \\ &= \alpha \circledcirc (1+\mu; \beta \circledcirc (1+\mu)).\end{aligned}$$

Finally, by using the compatibility of d with weighted braces, we obtain,

$$\begin{aligned}d(\alpha \circledcirc (1+\mu)) &= \sum_{n=0}^{+\infty} d(\alpha)\{\mu\}_n + (-1)^{|\alpha|} \sum_{n=1}^{+\infty} \alpha\{\mu, d(\mu)\}_{n-1,1} \\ &= d(\alpha) \circledcirc (1+\mu) + (-1)^{|\alpha|} \alpha \circledcirc (1+\mu; d(\mu)),\end{aligned}$$

which concludes the proof of the lemma. □

We can now prove assertion (ii) of Theorem A.

Theorem 1.2.29. *The gauge group G acts on the Maurer-Cartan set $\mathcal{MC}(L)$ by*

$$(1+\mu)\cdot\alpha = (\alpha + \mu\{\alpha\}_1 - d(\mu)) \circledcirc (1+\mu)^{\circledcirc -1}$$

for all $(1+\mu) \in G$ and $\alpha \in \mathcal{MC}(L)$.

Proof. We first need to prove that $\beta = (1+\mu)\cdot\alpha$ is indeed a Maurer-Cartan element. For this, we first remark that applying d on each side of the equality $d(\mu) = \alpha + \mu\{\alpha\}_1 - \beta \circledcirc (1+\mu)$, and by using that $d(\alpha) = -\alpha\{\alpha\}_1$ and the previous lemma, we have

$$d(\beta) \circledcirc (1+\mu) = -\alpha\{\alpha\}_1 - \mu\{\alpha\{\alpha\}_1\}_1 + d(\mu)\{\alpha\}_1 + \beta \circledcirc (1+\mu; d(\mu)).$$

Moreover, again by the previous lemma, we have

$$\begin{aligned}d(\mu)\{\alpha\}_1 &= \alpha\{\alpha\}_1 + \mu\{\alpha\}_1\{\alpha\}_1 - \beta \circledcirc (1+\mu)\{\alpha\}_1 \\ &= \alpha\{\alpha\}_1 + \mu\{\alpha\{\alpha\}_1\}_1 + \mu\{\alpha,\alpha\}_{1,1} - \beta \circledcirc (1+\mu;\alpha) - \beta \circledcirc (1+\mu; \mu\{\alpha\}_1).\end{aligned}$$

Then

$$d(\beta) \circledcirc (1+\mu) = \beta \circledcirc (1+\mu; d(\mu)) - \beta \circledcirc (1+\mu;\alpha) - \beta \circledcirc (1+\mu; \mu\{\alpha\}_1) + \mu\{\alpha,\alpha\}_{1,1}.$$

We note that $\mu\{\alpha,\alpha\}_{1,1} = 0$. Indeed, if we take the notations of the proof of Theorem 1.2.6, we have that $\mu\{\alpha,\alpha\}_{1,1} = \Gamma(\mathcal{P}re\mathcal{L}ie, \psi_{\mu,\alpha,\alpha})(\mathcal{O}F_2(e,e_1,e_2))$ where $|e| = 0$ and $|e_1| = |e_2| = 1$ are formal elements. We explicitly have that

$$\mathcal{O}F_2(e,e_1,e_2) = F_2(e,e_1,e_2) + ((12).F_2)(e_1,e,e_2) - ((13).F_2)(e_2,e_1,e)$$

$$-F_2(e,e_2,e_1) - ((12).F_2)(e_2,e,e_1) + ((13).F_2)(e_1,e_2,e).$$

Applying $\psi_{\mu,\alpha,\alpha}$ will then give $\mu\{\alpha,\alpha\}_{1,1} = 0$. We then finally have

$$d(\beta) \circledcirc (1+\mu) = -\beta \circledcirc (1+\mu; \beta \circledcirc (1+\mu)).$$

By the previous lemma, this gives

$$d(\beta) \circledcirc (1+\mu) = -\beta\{\beta\}_1 \circledcirc (1+\mu)$$

and then $(d(\beta) + \beta\{\beta\}_1) \circledcirc (1+\mu) = 0$, that is to say $d(\beta) + \beta\{\beta\}_1 = 0$ by composing with $(1+\mu)^{\circledcirc -1}$ on the right. We thus have proved that $\beta \in \mathcal{MC}(L)$.

We now need to check that we have indeed an action of G on $\mathcal{MC}(L)$. We have that $1+0$ acts trivially on $\mathcal{MC}(L)$, so we just need to prove that $((1+\nu) \circledcirc (1+\mu)) \cdot \alpha = (1+\nu) \cdot ((1+\mu) \cdot \alpha)$.

By hypothesis, we have the following equations:

$$d(\mu) = \alpha + \mu\{\alpha\}_1 - \beta \circledcirc (1+\mu),$$

$$d(\nu) = \beta + \nu\{\beta\}_1 - \gamma \circledcirc (1+\nu).$$

Let $1+\lambda = (1+\nu) \circledcirc (1+\mu) = 1 + \mu + \nu \circledcirc (1+\mu)$. We compute:

$$\begin{array}{rcl} \alpha + \lambda\{\alpha\}_1 - \gamma \circledcirc (1+\lambda) & = & \alpha + \mu\{\alpha\}_1 + \nu \circledcirc (1+\mu)\{\alpha\}_1 + d(\nu) \circledcirc (1+\mu) \\ & & -\beta \circledcirc (1+\mu) - \nu\{\beta\}_1 \circledcirc (1+\mu) \\ & & \\ & = & d(\mu) + d(\nu) \circledcirc (1+\mu) + \nu \circledcirc (1+\mu; \alpha) \\ & & +\nu \circledcirc (1+\mu; \mu\{\alpha\}_1) - \nu \circledcirc (1+\mu; \beta \circledcirc (1+\mu)) \end{array}$$

by the previous lemma. We then have

$$\begin{array}{rcl} \alpha + \lambda\{\alpha\}_1 - \gamma \circledcirc (1+\nu) \circledcirc (1+\mu) & = & d(\mu) + d(\nu) \circledcirc (1+\mu) + \nu \circledcirc (1+\mu; d(\mu)) \\ & = & d(\lambda), \end{array}$$

which proves the theorem. □

We end this section with the definition of the *Deligne groupoid*.

Proposition-Definition 1.2.30. *Let L be a complete $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra. We let* $\mathrm{Deligne}(L)$ *to be the category with* $\mathcal{MC}(L)$ *as set of objects and* $\mathrm{Mor}_{\mathrm{Deligne}(L)}(\alpha,\beta) = \{(1+\mu) \in G \mid (1+\mu) \cdot \alpha = \beta\}$.
Then $\mathrm{Deligne}(L)$ *is a groupoid called the* Deligne groupoid *of L.*

Proof. It is a corollary of the previous theorem. □

1.2.4 An integral Goldman-Millson theorem

We conclude this part with an analogue of the Goldman-Millson theorem. This theorem allows us to give a link between two particular groupoids when changing a dg Lie algebra L to another one $\overline{L}$ which is quasi-isomorphic to L (see [GM88, §2.4]).

Let A be a local artinian $\mathbf{K}$-algebra with maximal ideal $\mathfrak{m}_A$, where $\mathbf{K}$ is the field of fractions of some noetherian integral domain $\mathbb{K}$. Let L be a $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra (without any convergence hypothesis). If $\otimes$ denotes the tensor product over $\mathbb{K}$, then $L \otimes A$ is also a $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra with the following definitions:

$$\begin{array}{rcl}(L\otimes A)^k &=& L^k\otimes A,\\ \gamma(\mathcal{O}t(x_1\otimes a_1,\ldots,x_n\otimes a_n)) &=& \gamma(\mathcal{O}t(x_1,\ldots,x_n))\otimes a_1\cdots a_n,\\ d(x\otimes a) &=& d(x)\otimes a.\end{array}$$

To retrieve our convergence hypothesis, we can consider the sub $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra $L\otimes\mathfrak{m}_A$. This $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra has a filtration given by

$$F_n(L\otimes\mathfrak{m}_A)=L\otimes\mathfrak{m}_A^n$$

which is 0 for n big enough, because $\mathfrak{m}_A$ is nilpotent. In particular, our series will be reduced to finite sums.

Let $\mathrm{Deligne}(L,A)=\mathrm{Deligne}(L\otimes\mathfrak{m}_A)$ the associated Deligne groupoid. As in [GM88, §2.3], we remark that $\mathrm{Deligne}(-,-)$ defines a bifunctor such that, for all morphisms of $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebras $\varphi:L\longrightarrow\overline{L}$ and for all morphisms of algebras $\psi:A\longrightarrow\overline{A}$, we have the following diagram

$$\begin{tikzcd}\mathrm{Deligne}(L,A) \arrow[r, "\varphi_*"] \arrow[d, "\psi_*"'] & \mathrm{Deligne}(\overline{L},A) \arrow[d, "\psi_*"] \\ \mathrm{Deligne}(L,\overline{A}) \arrow[r, "\varphi_*"'] & \mathrm{Deligne}(\overline{L},\overline{A})\end{tikzcd}$$

which is commutative.

We can now prove Theorem B.

Theorem 1.2.31. *Let $\mathbb{K}$ be a noetherian integral domain and $\boldsymbol{K}$ its field of fractions. Let L and $\overline{L}$ be two positively graded $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebras. Let $\varphi:L\longrightarrow\overline{L}$ be a morphism of $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebras such that $H^0(\varphi)$ and $H^1(\varphi)$ are isomorphisms, and $H^2(\varphi)$ is a monomorphism. Then for every local artinian $\boldsymbol{K}$-algebras A, the induced functor $\varphi_*:\mathrm{Deligne}(L,A)\longrightarrow\mathrm{Deligne}(\overline{L},A)$ is an equivalence of groupoids.*

Proof. The proof is close to the one given in [GM88, §2.5-§2.11]. By the same arguments as in [GM88, §2.5], we are reduced to prove the following. Let A be a local artinian $\mathbf{K}$-algebra with maximal ideal $\mathfrak{m}_A$ and $\mathfrak{I}\subset A$ be an ideal such that $\mathfrak{I}\cdot\mathfrak{m}_A=0$. Then if the theorem holds for $A/\mathfrak{I}$, then it also holds for A.

Our first goal is to prove the same proposition given in [GM88, §2.6], which constructs three obstruction maps o_2,o_1 and o_0. Let $\pi_*:L\otimes A\longrightarrow L\otimes A/\mathfrak{I}$ be the map induced by the canonical projection $\pi:A\longrightarrow A/\mathfrak{I}$.

We first define $o_2:Obj(\mathrm{Deligne}(L,A/\mathfrak{I}))\longrightarrow H^2(L\otimes\mathfrak{I})$ which is such that $o_2(\omega)=0$ if and only if there exists $\widetilde{\omega}\in Obj(\mathrm{Deligne}(L,A))$ with $\pi_*(\widetilde{\omega})=\omega$.

Let $\omega\in Obj(\mathrm{Deligne}(L,A/\mathfrak{I}))\subset\mathfrak{m}_A/\mathfrak{I}$. Let $\widetilde{\omega}\in L^1\otimes\mathfrak{m}_A$ be such that $\pi_*(\widetilde{\omega})=\omega$ and $\mathcal{Q}(\widetilde{\omega})=d(\widetilde{\omega})+\widetilde{\omega}\{\widetilde{\omega}\}_1$. Then $\pi_*(\mathcal{Q}(\widetilde{\omega}))=0$ to that $\mathcal{Q}(\widetilde{\omega})\in L^2\otimes\mathfrak{I}$. By using that $\mathfrak{I}\cdot\mathfrak{m}_A=0$, we obtain

$$\begin{array}{rcl}d(\mathcal{Q}(\widetilde{\omega})) &=& d(\widetilde{\omega})\{\widetilde{\omega}\}_1-\widetilde{\omega}\{d(\widetilde{\omega})\}_1\\ &=& -\widetilde{\omega}\{\widetilde{\omega}\}_1\{\widetilde{\omega}\}_1+\widetilde{\omega}\{\widetilde{\omega}\{\widetilde{\omega}\}_1\}_1\\ &=& \widetilde{\omega}\{\widetilde{\omega},\widetilde{\omega}\}_{1,1}\\ &=& 0.\end{array}$$

This implies that $\mathcal{Q}(\widetilde{\omega}) \in Z^2(L\otimes\mathfrak{I})$. Let $\widetilde{\omega}' \in L^1\otimes\mathfrak{m}_A$ be some other element such that $\pi_*(\widetilde{\omega}') = \omega$. In particular, $\widetilde{\omega}-\widetilde{\omega}' \in L^1\otimes\mathfrak{I}$. We then have, using again that $\mathfrak{I}\cdot\mathfrak{m}_A = 0$,

$$\begin{aligned}\mathcal{Q}(\widetilde{\omega}')-\mathcal{Q}(\widetilde{\omega}) &= d(\widetilde{\omega}'-\widetilde{\omega}) + (\widetilde{\omega}'-\widetilde{\omega})\{\widetilde{\omega}'-\widetilde{\omega}\}_1 \\ &\quad +\widetilde{\omega}\{\widetilde{\omega}'-\widetilde{\omega}\}_1 + (\widetilde{\omega}'-\widetilde{\omega})\{\widetilde{\omega}\}_1 \\ &= d(\widetilde{\omega}'-\widetilde{\omega}).\end{aligned}$$

We then let $o_2(\omega)$ to be the class of $\mathcal{Q}(\widetilde{\omega})$ in $H^2(L\otimes\mathfrak{I})$. Suppose that $o_2(\omega)=0$. Then by definition, there exists some $\psi \in L^1\otimes\mathfrak{I}$ such that $\mathcal{Q}(\widetilde{\omega}) = d(\psi)$. We can check that $\widetilde{\omega}' := \widetilde{\omega}-\psi \in Obj(\mathrm{Deligne}(L,A))$ and $\pi_*(\widetilde{\omega}') = \omega$. In the converse direction, we obviously have that if $\omega = \pi_*(\widetilde{\omega})$ with $\widetilde{\omega} \in Obj(\mathrm{Deligne}(L,A))$, then $o_2(\omega)=0$.

We now prove the following analogue of the lemma given in [GM88, §2.8]. For all $\alpha \in L^1\otimes\mathfrak{m}_A, \eta \in L^0\otimes\mathfrak{m}_A$ and $u \in L^0\otimes\mathfrak{I}$, we have

$$(1+u+\eta)\cdot\alpha = (1+\eta)\cdot\alpha - d(u).$$

Let $\beta = (1+\eta)\cdot\alpha$. We have

$$\begin{aligned}(\beta-d(u))\circledcirc(1+u+\eta) &= \sum_{n=0}^{+\infty}(\beta-d(u))\{u+\eta\}_n \\ &= \sum_{n=0}^{+\infty}\sum_{k=0}^{n}(\beta-d(u))\{u,\eta\}_{k,n-k}.\end{aligned}$$

Since $\mathfrak{I}\cdot\mathfrak{m}_A=0$, and because $u \in L^1\otimes\mathfrak{I}$ and $\eta\in L^0\otimes\mathfrak{m}_A$, the terms with $n\neq 0$ and $k\neq 0$ are 0 by definition of the $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra structure in $L\otimes A$. We then have

$$\begin{aligned}(\beta-d(u))\circledcirc(1+u+\eta) &= \beta - d(u) + \sum_{n=1}^{+\infty}\beta\{\eta\}_n \\ &= \beta\circledcirc(1+\eta)-d(u) \\ &= \alpha+\eta\{\alpha\}_1 - d(\eta)-d(u) \\ &= \alpha+(u+\eta)\{\alpha\}_1 - d(u+\eta)\end{aligned}$$

which finally gives

$$(1+\eta)\cdot\alpha - d(u) = (\alpha+(u+\eta)\{\alpha\}_1-d(u+\eta))\circledcirc(1+u+\eta)^{\circledcirc-1} = (1+u+\eta)\cdot\alpha.$$

Let $\xi \in Obj(\mathrm{Deligne}(L,A/\mathfrak{I}))$. We let $\pi_*^{-1}(\xi)$ to be the category whose objects are elements $\omega\in Obj(\mathrm{Deligne}(L,A))$ such that $\pi_*(\omega)=\xi$, and with morphisms the gauge group elements $\gamma\in\mathrm{Deligne}(L,A)$ such that $\pi_*(\gamma)=1$. We now construct a map $o_1 : Obj(\pi_*^{-1}(\xi))\times Obj(\pi_*^{-1}(\xi)) \longrightarrow Z^1(L\otimes\mathfrak{I})$ such that $o_1(\alpha,\beta)=0$ if and only if there exists a morphism γ in $\pi_*^{-1}(\xi)$ such that $\gamma(\alpha)=\beta$.

Let $\eta\in Z^1(L\otimes\mathfrak{I})$. For every $\alpha\in Obj(\pi_*^{-1}(\xi))$, we have that $\mathcal{Q}(\alpha+\eta) = \mathcal{Q}(\alpha)+\alpha\{\eta\}_1+\eta\{\eta\}_1=0$, since $\mathfrak{I}\cdot\mathfrak{m}_A=0$. We then have that $\alpha+\eta\in Obj(\mathrm{Deligne}(L,A))$, and $\pi_*(\alpha+\eta)=\xi$. We thus have an action of the group $Z^1(L\otimes\mathfrak{I})$ on the set $Obj(\pi_*^{-1}(\xi))$. This action is simply transitive. Indeed, let $\alpha,\beta\in Obj(\mathrm{Deligne}(L,A))$ be such that $\pi_*(\alpha)=\pi_*(\beta)=\xi$. Then $\alpha-\beta\in L^1\otimes\mathfrak{I}$ and $d(\alpha-\beta)=\mathcal{Q}(\alpha)-\mathcal{Q}(\beta)=0$. The element $\eta:=\alpha-\beta$ is then an element of $Z^1(L\otimes\mathfrak{I})$ whose action on β is α. Since

$\alpha - \beta = 0$ if and only if $\alpha = \beta$, the action is indeed simply transitive. We then can set $o_1(\alpha, \beta)$ to be the class of $\alpha - \beta$ in $H^1(L \otimes \mathfrak{I})$.

Now, if there exists some element $1 + u \in G$ in the gauge group of $L \otimes \mathfrak{m}_A$ such that $(1+u)\cdot\alpha = \beta$ and $\pi_*(1+u) = 1$, we then have, according to the analogue of the lemma given in [GM88, §2.8], that $\beta = (1+u)\cdot\alpha = \alpha - d(u)$ so that $o_1(\alpha, \beta) = 0$. In the converse direction, if $o_1(\alpha, \beta) = 0$, then there exists $u \in L^0 \otimes \mathfrak{I}$ such that $\alpha - \beta = d(u)$. By using the lemma again, we find that $(1+u)\cdot\alpha = \beta$.

Let $\widetilde{\alpha}, \widetilde{\beta} \in Obj(\mathrm{Deligne}(L, A))$ and $\alpha = \pi_*(\widetilde{\alpha}), \beta = \pi_*(\widetilde{\beta})$ be such that there exists an element $1+u$ of the gauge group of $L\otimes\mathfrak{m}_A/\mathfrak{I}$ with $(1+u)\cdot\alpha = \beta$. Let $\pi_*^{-1}(1+u)$ be the set of gauge group elements $1+\widetilde{u}$ in $L\otimes\mathfrak{m}_A$ such that $(1+\widetilde{u})\cdot\widetilde{\alpha} = \widetilde{\beta}$ and $\pi_*(1+\widetilde{u}) = 1+u$. We finally construct a map $o_0 : \pi_*^{-1}(1+u) \times \pi_*^{-1}(1+u) \longrightarrow H^0(L \otimes \mathfrak{I})$ which satisfies the following. For every $1+\widetilde{u}, 1+\widetilde{u}' \in \pi_*^{-1}(1+u)$, we have that $o_0(1+\widetilde{u}, 1+\widetilde{u}') = 0$ if and only if $\widetilde{u} = \widetilde{u}'$.

We define a simply transitive action of $H^0(L \otimes \mathfrak{I})$ on the set $\pi_*^{-1}(1+u)$. Let $1+\widetilde{u} \in \pi_*^{-1}(1+u)$ and let $w \in L \otimes \mathfrak{I}$. We have that

$$(1+\widetilde{u}+w)\cdot\widetilde{\alpha} = (1+\widetilde{u})\cdot\widetilde{\alpha} - d(w) = \widetilde{\beta} - d(w).$$

Therefore, if $w \in H^0(L \otimes \mathfrak{I}) = Z^0(L \otimes \mathfrak{I})$, then $(1+\widetilde{u}+w)\cdot\widetilde{\alpha} = \widetilde{\beta}$. This then defines an action of $H^0(L \otimes \mathfrak{I})$ on $\pi_*^{-1}(1+u)$. Let $1+\widetilde{u}' \in \pi_*^{-1}(1+u)$. We set $w = o_0(1+\widetilde{u}, 1+\widetilde{u}') := \widetilde{u} - \widetilde{u}' \in L \otimes \mathfrak{I}$. Then

$$d(w) = d(o_0(1+\widetilde{u}, 1+\widetilde{u}')) = (1+\widetilde{u}')\cdot\widetilde{\alpha} - (1+\widetilde{u}'+w)\cdot\widetilde{\alpha} = \widetilde{\beta} - \widetilde{\beta} = 0.$$

We then obtain that $w \in H^0(L \otimes \mathfrak{I})$ is the unique element which sends $1+\widetilde{u}$ to $1+\widetilde{u}'$. The action is then simply transitive, and we have constructed o_0.

This then proves an analogue of the proposition given in [GM88, §2.6]. The other parts of the proof only use these three obstructions maps and do not directly use the structure of L. We can then follow exactly the same arguments in [GM88, §2.11] to obtain the result.

□

Definition 1.2.32. *Two positively graded $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebras L and $\overline{L}$ are* quasi-isomorphic *if there exists a zig-zag of morphisms of $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebras*

$$L = L_0 \longrightarrow L_1 \longleftarrow \cdots \longrightarrow L_{m-1} \longleftarrow L_m = \overline{L}$$

in which each morphism induces an isomorphism in cohomology.

Corollary 1.2.33. *If L and $\overline{L}$ are quasi-isomorphic, then for all local artinian $\boldsymbol{K}$-algebras A, the groupoids* $\mathrm{Deligne}(L, A)$ *and* $\mathrm{Deligne}(\overline{L}, A)$ *are equivalent. More precisely, we have a zig-zag of equivalence of groupoids*

$$\mathrm{Deligne}(L, A) \longrightarrow \mathrm{Deligne}(L_1, A) \longleftarrow \cdots \longrightarrow \mathrm{Deligne}(L_{m-1}, A) \longleftarrow \mathrm{Deligne}(\overline{L}, A)$$

which is natural in A.

1.3 Application in homotopy theory for operads

The goal of this section is to establish Theorem C, which gives a computation of $\pi_0(\mathrm{Map}(B^c(\mathcal{C}),\mathcal{P}))$ where $\mathcal{C}$ is a Σ_*-cofibrant coaugmented cooperad, $\mathcal{P}$ an augmented operad and B^c the cobar construction (see [Fre17b] or [LV12] for a definition of this construction). In the case of a field of characteristic 0, it can be expressed in terms of the Deligne groupoid with the structure of dg Lie algebra of $\mathrm{Hom}_\Sigma(\overline{\mathcal{C}},\overline{\mathcal{P}})$. We extend this result using a structure of $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra that underlies this dg Lie algebra structure.

In §1.3.1, we define infinitesimal k-compositions and k-decompositions that generalize the usual infinitesimal composition and decomposition operations given in [LV12, §6.1]. These operations will be used in the next section to write more easily weighted brace operations of the convolution operad.

In §1.3.2, we recall the definition of the convolution operad $\mathrm{Hom}(\mathcal{C},\mathcal{P})$, as given in [LV12, §6.4.1], and study the $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra structure of $\mathrm{Hom}_\Sigma(\mathcal{C},\mathcal{P})$. In the same way that infinitesimal composition and decomposition can be used to express the pre-Lie algebra structure of the convolution operad (see [LV12, Proposition 6.4.5]), we will use infinitesimal k-compositions and k-decompositions to compute weighted brace operations of the convolution operad.

In §1.3.3, we just use a cylinder object of $B^c(\mathcal{C})$ given by Fresse in [Fre09b, §5.1] to get our result: the quotient of $\mathrm{Hom}_\Sigma(\mathcal{C},\mathcal{P})$ by the gauge action gives $\pi_0(\mathrm{Map}(B^c(\mathcal{C}),\mathcal{P}))$.

1.3.1 Infinitesimal compositions and decompositions of an operad and a cooperad

We first introduce some definitions which will be useful for the computations.

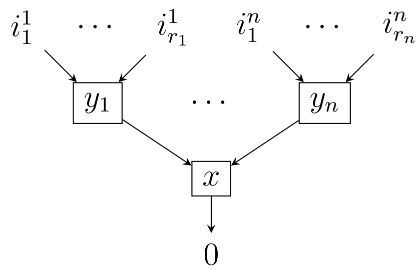
Let M and N be two symmetric sequences such that $N(0)=0$. Recall that we have a monoidal structure on the category of symmetric sequences defined by

$$M\circ N(n)=\bigoplus_{k\geq 0}M(k)\otimes_{\Sigma_k}\left(\bigoplus_{i_1+\cdots+i_k=n}Ind_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_k}}^{\Sigma_n}(N(i_1)\otimes\cdots\otimes N(i_k))\right),$$

with as unit the symmetric sequence I defined by

$$I(n)=\begin{cases}\mathbb{K} & \text{if } n=1\\ 0 & \text{if } n\neq 1\end{cases}.$$

Every elements of $M\circ N(n)$ can be identified as a tree of the form:



,

where $x \in M(n)$, $y_1 \in N(r_1), \dots, y_n \in N(r_n)$, and where $1 \le i_p^q \le \sum_i r_i$ are labels which represent a permutation of $\Sigma_{r_1+\dots+r_n}$.

Note that we can write $M \circ N(n)$ without quotients by the group permutations by taking a choice of set of representatives. This set is given by pointed shuffle permutations (see conventions):

$$M \circ N(n) = \bigoplus_{k \ge 0} M(k) \otimes \left(\bigoplus_{i_1+\dots+i_k=n} N(i_1) \otimes \dots \otimes N(i_k) \otimes \mathbb{K}[Sh_*(i_1, \dots, i_k)] \right).$$

We now generalize the definition of the infinitesimal composition/decomposition defined in [LV12], in order to write some formulas in a more convenient way.

Definition 1.3.1. *Let M and N be two symmetric sequences. For all $k \ge 0$, we define a new symmetric sequence denoted by $M \circ_{(k)} N$ called the* k-infinitesimal composite *of M and N defined, in each arity n, as the submodule of $M \circ (I \oplus N)(n)$ spanned by trees where exactly k elements at level 2 are in N, and the others in I.*

Let M, M', N and N' be symmetric sequences. One can easily check that if we have morphisms of symmetric sequences $f : M \longrightarrow M'$ and $g : N \longrightarrow N'$, then we have a morphism $f \circ_{(k)} g : M \circ_{(k)} N \longrightarrow M' \circ_{(k)} N'$ induced by $f \circ (id_I \oplus g)$.

Let $\mathcal{P}$ be an operad with composition $\gamma : \mathcal{P} \circ \mathcal{P} \longrightarrow \mathcal{P}$ and unit $\eta : I \longrightarrow \mathcal{P}$, and let $\mathcal{C}$ a cooperad with coproduct $\Delta : \mathcal{C} \longrightarrow \mathcal{C} \circ \mathcal{C}$ and counit $\varepsilon : \mathcal{C} \longrightarrow I$. We will suppose that $\mathcal{P}$ is augmented, i.e. the unit $\eta : I \longrightarrow \mathcal{P}$ admits a retraction $\pi : \mathcal{P} \longrightarrow I$. We then have that there exists a symmetric sequence $\overline{\mathcal{P}}$ with $\mathcal{P} \simeq I \oplus \overline{\mathcal{P}}$ such that the first projection on $\mathcal{P}$ is given by π. Similarly, we suppose that $\mathcal{C}$ is coaugmented, i.e. the counit $\varepsilon : \mathcal{C} \longrightarrow I$ admits a section $s : I \longrightarrow \mathcal{C}$. We then have that there exists a symmetric sequence $\overline{\mathcal{C}}$ with $\mathcal{C} \simeq I \oplus \overline{\mathcal{C}}$ such that the first projection is given by ε.

In the following, we assume that $\mathcal{C}(0) = \mathcal{P}(0) = 0$ and $\mathcal{C}(1) = \mathcal{P}(1) = \mathbb{K}$.

We give an extension of the usual infinitesimal composition and decomposition operations given in [LV12, §6.1] for $k = 1$.

Definition 1.3.2. *Let $k \ge 1$.*

- *We define* the infinitesimal k-composition *in $\mathcal{P}$ as*

$$\gamma_{(k)} : \overline{\mathcal{P}} \circ_{(k)} \overline{\mathcal{P}}(n) \longrightarrow \mathcal{P} \circ \mathcal{P}(n) \xrightarrow{\gamma} \mathcal{P}(n),$$

where the first map is the inclusion of $\overline{\mathcal{P}} \circ_{(k)} \overline{\mathcal{P}}$ in $\mathcal{P} \circ \mathcal{P}$.

- *We define the* infinitesimal k-decomposition *in* $\mathcal{C}$ *as*

$$\Delta_{(k)} : \mathcal{C}(n) \xrightarrow{\Delta} \mathcal{C} \circ \mathcal{C}(n) \longrightarrow \overline{\mathcal{C}} \circ_{(k)} \overline{\mathcal{C}}(n),$$

where the last map is the projection of $\mathcal{C} \circ \mathcal{C}$ *onto* $\overline{\mathcal{C}} \circ_{(k)} \overline{\mathcal{C}}$.

Because $\mathcal{C}$ is coaugmented, we have that the coproduct $\Delta : \mathcal{C} \longrightarrow \mathcal{C} \circ \mathcal{C}$ preserves the isomorphism $\mathcal{C} \simeq I \oplus \overline{\mathcal{C}}$ in the following sense. We have the isomorphism

$$\mathcal{C} \circ \mathcal{C} \simeq I \circ I \oplus \overline{\mathcal{C}} \circ I \oplus I \circ \overline{\mathcal{C}} \oplus \bigoplus_{k \geq 1} \overline{\mathcal{C}} \circ_{(k)} \overline{\mathcal{C}}.$$

Then, we get that the restriction of Δ on I and on $\overline{\mathcal{C}}$ are such that $\Delta : I \longrightarrow I \circ I$ and $\Delta : \overline{\mathcal{C}} \longrightarrow \overline{\mathcal{C}} \circ I \oplus I \circ \overline{\mathcal{C}} \oplus \bigoplus_{k \geq 1} \overline{\mathcal{C}} \circ_{(k)} \overline{\mathcal{C}}$. We can then define the infinitesimal k-decompositions on $\overline{\mathcal{C}}$ by

$$\Delta_{(0)} : \overline{\mathcal{C}} \xrightarrow{\Delta} \overline{\mathcal{C}} \circ I \oplus I \circ \overline{\mathcal{C}} \oplus \bigoplus_{k \geq 1} \overline{\mathcal{C}} \circ_{(k)} \overline{\mathcal{C}} \twoheadrightarrow \overline{\mathcal{C}} \circ I \oplus I \circ \overline{\mathcal{C}},$$

$$\Delta_{(k)} : \overline{\mathcal{C}} \xrightarrow{\Delta} \overline{\mathcal{C}} \circ I \oplus I \circ \overline{\mathcal{C}} \oplus \bigoplus_{k \geq 1} \overline{\mathcal{C}} \circ_{(k)} \overline{\mathcal{C}} \twoheadrightarrow \overline{\mathcal{C}} \circ_{(k)} \overline{\mathcal{C}},$$

for all $k \geq 1$.

1.3.2 $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra structure of the convolution operad

Let M and N be two symmetric sequences of differential graded $\mathbb{K}$-modules. We define a new symmetric sequence $\mathrm{Hom}(M,N)$ in dg $\mathbb{K}$-modules by $\mathrm{Hom}(M,N)(n) = \mathrm{Hom}(M(n), N(n))$, the differential graded module formed by the homogeneous morphisms $f : M(n) \longrightarrow N(n)$. The differential on $\mathrm{Hom}(M,N)$ is given by

$$d(f) = d_M \circ f - (-1)^{deg(f)} f \circ d_N,$$

for all $f \in \mathrm{Hom}(M,N)$. The action of Σ_n on $\mathrm{Hom}(M(n), N(n))$ is defined by

$$\forall x \in M(n), (\sigma \cdot f)(x) = \sigma \cdot f(\sigma^{-1} \cdot x),$$

for all $\sigma \in \Sigma_n$.

Proposition 1.3.3 (see [LV12]). *Let* $\mathcal{C}$ *be a cooperad and* $\mathcal{P}$ *be an operad. Then* $\mathrm{Hom}(\mathcal{C}, \mathcal{P})$ *has the structure of a dg operad called the* convolution operad *of* $\mathcal{C}$ *and* $\mathcal{P}$.

We recall the operad structure on $\mathrm{Hom}(\mathcal{C}, \mathcal{P})$. For $f \in \mathrm{Hom}(\mathcal{C}, \mathcal{P})(k)$, $g_1 \in \mathrm{Hom}(\mathcal{C}, \mathcal{P})(i_1), \ldots, g_k \in \mathrm{Hom}(\mathcal{C}, \mathcal{P})(i_k)$ the composition $\gamma(f \otimes g_1 \otimes \cdots \otimes g_k)$ is given by the composite

$$\begin{array}{ccccc} \mathcal{C}(n) \xrightarrow{\Delta} \mathcal{C}\circ\mathcal{C}(n) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{C}(k)\otimes\mathcal{C}(i_1)\otimes\cdots\otimes\mathcal{C}(i_k)\otimes\mathbb{K}[id] \\ \big\downarrow{\scriptstyle f\otimes g_1\otimes\cdots\otimes g_k\otimes id} \\ \mathcal{P}(k)\otimes\mathcal{P}(i_1)\otimes\cdots\otimes\mathcal{P}(i_k)\otimes\mathbb{K}[id] \hookrightarrow \mathcal{P}\circ\mathcal{P}(n) \xrightarrow{\gamma} \mathcal{P}(n) \end{array}$$

where $n = \sum_p i_p$.

We have $\mathrm{Hom}_{\Sigma_n}(\mathcal{C}(n),\mathcal{P}(n)) = \mathrm{Hom}(\mathcal{C}(n),\mathcal{P}(n))^{\Sigma_n}$, and we set

$$\mathrm{Hom}_\Sigma(\mathcal{C},\mathcal{P}) = \prod_{n\geq 1} \mathrm{Hom}_{\Sigma_n}(\mathcal{C}(n),\mathcal{P}(n));$$

$$\mathrm{Hom}_\Sigma(\overline{\mathcal{C}},\overline{\mathcal{P}}) = \prod_{n\geq 2} \mathrm{Hom}_{\Sigma_n}(\mathcal{C}(n),\mathcal{P}(n)) \subset \mathrm{Hom}_\Sigma(\mathcal{C},\mathcal{P}).$$

Then, according to Corollary 1.2.18, we have that $\mathrm{Hom}_\Sigma(\overline{\mathcal{C}},\overline{\mathcal{P}})$ is endowed with a complete $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra structure. We also have the isomorphism

$$\mathrm{Hom}_\Sigma(\mathcal{C},\mathcal{P}) \simeq \mathbb{K}\oplus \mathrm{Hom}_\Sigma(\overline{\mathcal{C}},\overline{\mathcal{P}}),$$

so that any morphism in $\mathrm{Hom}_\Sigma(\overline{\mathcal{C}},\overline{\mathcal{P}})$ can be identified with a morphism in $\mathrm{Hom}_\Sigma(\mathcal{C},\mathcal{P})$ which is 0 on $\mathcal{C}(1)=\mathbb{K}$.

We can explicitly describe the weighted braces with one input of $\mathrm{Hom}_\Sigma(\overline{\mathcal{C}},\overline{\mathcal{P}})$ in terms of infinitesimal decompositions and compositions.

Lemma 1.3.4. *Let $\overline{f},\overline{g}\in \mathrm{Hom}_\Sigma(\overline{\mathcal{C}},\overline{\mathcal{P}})$. Then $\overline{f}\{\overline{g}\}_k$ is given by the composite*

$$\overline{\mathcal{C}} \xrightarrow{\Delta_{(k)}} \overline{\mathcal{C}}\circ_{(k)}\overline{\mathcal{C}} \xrightarrow{\overline{f}\circ_{(k)}\overline{g}} \overline{\mathcal{P}}\circ_{(k)}\overline{\mathcal{P}} \xrightarrow{\gamma_{(k)}} \overline{\mathcal{P}}\ .$$

In particular, we retrieve the well-known pre-Lie algebra structure on $\mathrm{Hom}_\Sigma(\overline{\mathcal{C}},\overline{\mathcal{P}})$ given by the composite

$$\overline{\mathcal{C}} \xrightarrow{\Delta_{(1)}} \overline{\mathcal{C}}\circ_{(1)}\overline{\mathcal{C}} \xrightarrow{\overline{f}\circ_{(1)}\overline{g}} \overline{\mathcal{P}}\circ_{(1)}\overline{\mathcal{P}} \xrightarrow{\gamma_{(1)}} \overline{\mathcal{P}}$$

as shown in [LV12, Proposition 6.4.5] (note that here we consider left actions).

Proof. By definition of the $\Gamma(\mathcal{P}re\mathcal{L}ie,-)-$algebra structure of $\mathrm{Hom}_\Sigma(\overline{\mathcal{C}},\overline{\mathcal{P}})$ (see Corollary 1.2.18), it is sufficient to prove it on $\bigoplus_{r\geq 2}\mathrm{Hom}_{\Sigma_r}(\mathcal{C}(r),\mathcal{P}(r))$. We write $\overline{g} = \overline{g_1}+\cdots+\overline{g_p}$ where $\overline{g_i}\in \mathrm{Hom}_{\Sigma_{n_i}}(\mathcal{C}(n_i),\mathcal{P}(n_i))$ with $n_i\neq n_j$ whenever $i\neq j$. Since the identity we need to prove is linear in $\overline{f}$, we can suppose that $\overline{f}\in \mathrm{Hom}_{\Sigma_n}(\mathcal{C}(n),\mathcal{P}(n))$. We then have that

$$\overline{f}\{\overline{g}\}_k = \sum_{r_1+\cdots+r_p=k} \overline{f}\{\overline{g_1},\dots,\overline{g_p}\}_{r_1,\dots,r_p}.$$

If we denote by γ the operadic composition in the operad $\mathrm{Hom}(\mathcal{C},\mathcal{P})$, and if we set $\overline{h_1},\dots,\overline{h_k} = \underbrace{\overline{g_1},\dots,\overline{g_1}}_{r_1},\dots,\underbrace{\overline{g_p},\dots,\overline{g_p}}_{r_p}$ and $s_1,\dots,s_k = \underbrace{n_1,\dots,n_1}_{r_1},\dots,\underbrace{n_p,\dots,n_p}_{r_p}$, then by definition of the $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra structure of $\bigoplus_{r\geq 2}\mathrm{Hom}_{\Sigma_r}(\mathcal{C}(r),\mathcal{P}(r))$ (see Proposition 1.2.17), we have

$$\overline{f}\{\overline{g_1},\ldots,\overline{g_p}\}_{r_1,\ldots,r_p} = \sum_{\sigma\in Sh(r_1,\ldots,r_p)} \sum_{\substack{1\leq i_1<\cdots<i_k\leq n\\ \omega\in Sh_*(1,\ldots,\underset{i_1}{s_{\sigma^{-1}(1)}},\ldots,\underset{i_k}{s_{\sigma^{-1}(k)}},\ldots,1)}} \pm\omega\cdot\gamma(\overline{f}(1,\ldots,\underset{i_1}{\overline{g}_{\sigma^{-1}(1)}},\ldots,\underset{i_k}{\overline{g}_{\sigma^{-1}(k)}},\ldots,1)).$$

For a given $\sigma\in Sh(r_1,\ldots,r_p)$ and $\omega\in Sh_*(1,\ldots,\underset{i_1}{s_{\sigma^{-1}(1)}},\ldots,\underset{i_k}{s_{\sigma^{-1}(k)}},\ldots,1)$, we can see the corresponding term in the sum as the composite

$$\begin{array}{l} \mathcal{C}(m)\xrightarrow{\Delta}\mathcal{C}\circ\mathcal{C}(m)\twoheadrightarrow\mathcal{C}(n)\otimes\mathcal{C}(s_{\sigma^{-1}(1)})\otimes\cdots\otimes\mathcal{C}(s_{\sigma^{-1}(k)})\otimes\mathbb{K}[\omega] \\ \qquad\qquad\qquad\qquad \big\downarrow{\scriptstyle \overline{f}\otimes\overline{h}_{\sigma^{-1}(1)}\otimes\cdots\otimes\overline{h}_{\sigma^{-1}(k)}\otimes id} \\ \mathcal{P}(n)\otimes\mathcal{P}(s_{\sigma^{-1}(1)})\otimes\cdots\otimes\mathcal{P}(s_{\sigma^{-1}(k)})\otimes\mathbb{K}[\omega]\hookrightarrow\mathcal{P}\circ\mathcal{P}(m) \\ \qquad\qquad\qquad\qquad \xrightarrow{\qquad\gamma\qquad}\mathcal{P}(m) \end{array}$$

where $m=n+(n_1-1)r_1+\cdots+(n_p-1)r_p$, and where we omits some unit elements. Summing over the ω's and using the decomposition of $\mathcal{C}\circ\mathcal{C}(m)$ and $\mathcal{P}\circ\mathcal{P}(m)$ just before Definition 1.3.1 will then give the composite

$$\mathcal{C}(m)\xrightarrow{\Delta}\mathcal{C}\circ\mathcal{C}(m)\xrightarrow{\overline{f}\otimes\overline{h}_{\sigma^{-1}(1)}\otimes\cdots\otimes\overline{h}_{\sigma^{-1}(k)}}\mathcal{P}\circ\mathcal{P}(m)\xrightarrow{\gamma}\mathcal{P}(m)$$

where we omit, again, the unit elements. Note that, since $\overline{f}$ and the $\overline{h}_i$'s are 0 on $\mathcal{C}(1)$, this composite is also 0 on $\mathcal{C}(1)$. Summing on all $\sigma\in Sh(r_1,\ldots,r_p)$ will give the desired composite. □

Theorem 1.3.5. *The circular product of two elements $f=1+\overline{f}, g=1+\overline{g}$ of the gauge group of* $\mathrm{Hom}_\Sigma(\overline{\mathcal{C}},\overline{\mathcal{P}})$ *is given by*

$$f\circledcirc g:\mathcal{C}\xrightarrow{\Delta}\mathcal{C}\circ\mathcal{C}\xrightarrow{f\circ g}\mathcal{P}\circ\mathcal{P}\xrightarrow{\gamma}\mathcal{P}\ .$$

Proof. Because $f_{|I}=g_{|I}=1$, we have that $(f\circledcirc g)_{|I}=1$. We thus need to show the equality on $\overline{\mathcal{C}}$. Recall that we have infinitesimal decompositions on $\overline{\mathcal{C}}$ denoted by $\Delta_{(0)}$ and $\Delta_{(k)}$ for $k\geq 1$ such that $\Delta_{|\overline{\mathcal{C}}}=\Delta_{(0)}\oplus\bigoplus_{k\geq 1}\Delta_{(k)}$. The map $\Delta_{(0)}$ will give $\overline{f}+\overline{g}$, and each $\Delta_{(k)}$ will give $\overline{f}\{\overline{g}\}_k$ according to the previous lemma. We thus have that the composite in the statement of the theorem gives

$$1+\overline{f}+\sum_{k\geq 0}\overline{f}\{\overline{g}\}_k$$

which is exactly $f\circledcirc g$. □

1.3.3 Computation of $\pi_0(\mathrm{Map}(B^c(\mathcal{C}),\mathcal{P}))$

We now extend the computation of $\pi_0(\mathrm{Map}(B^c(\mathcal{C}),\mathcal{P}))$ on a field $\mathbb{K}$ with positive characteristic. In this last section, we chose to work with a homological convention to follow the conventions in the literature. Note that this does not change anything on the results of the previous sections.

Recall that we can give an explicit cylinder object for $B^c(\mathcal{C})$, where B^c is the cobar construction of $\mathcal{C}$, when $\mathcal{C}$ is Σ_*-cofibrant (see for instance [Fre09b] or [LV12]). Explicitly, let $K=\mathbb{K}\sigma^0\oplus\mathbb{K}\sigma^1\oplus\mathbb{K}\sigma^{01}$ where $|\sigma^0|=|\sigma^1|=-1$, $|\sigma^{01}|=0$ and $d(\sigma^{01})=\sigma^1-\sigma^0$. Then there exists a derivation of operads ∂ such that the free dg operad $(\mathcal{F}(K\otimes\overline{\mathcal{C}}),\partial)$ is a cylinder object for $B^c(\mathcal{C})$. We refer to [Fre09b, §5.1] for an explicit construction of ∂ and a proof of the previous statement.

Theorem 1.3.6. *Suppose that $\mathcal{C}$ is Σ_*-cofibrant. We then have a bijection:*

$$\pi_0(\mathrm{Map}(B^c(\mathcal{C}),\mathcal{P}))\simeq\pi_0\mathrm{Deligne}(\mathrm{Hom}_\Sigma(\overline{\mathcal{C}},\overline{\mathcal{P}})).$$

Proof. Recall from Fresse (see [Fre17b, Theorem 3.2.14] for instance) that $\pi_0(\mathrm{Map}(B^c(\mathcal{C}),\mathcal{P}))\simeq(\mathrm{Mor}(B^c(\mathcal{C}),\mathcal{P}),\sim_h)$ where $\sim_h$ is the homotopy relation in the category of symmetric operads. Recall also that the data of a Maurer-Cartan element α in $\mathrm{Hom}_\Sigma(\overline{\mathcal{C}},\overline{\mathcal{P}})$ is equivalent to give a morphism of operads ϕ_α from $B^c(\mathcal{C})$ to $\mathcal{P}$ (see [Fre09b] or [LV12]). We just need to show that the action of the gauge group on the Maurer-Cartan set of $\mathrm{Hom}_\Sigma(\overline{\mathcal{C}},\overline{\mathcal{P}})$ from one Maurer-Cartan α to an other one β is equivalent to give a homotopy from ϕ_α to ϕ_β.

Let $1+\lambda$ be an element of the gauge group. We define a morphism $h:\mathrm{Cyl}(B^c(\mathcal{C}))\longrightarrow\mathcal{P}$ via $h:K\otimes\overline{\mathcal{C}}\longrightarrow\mathcal{F}(K\otimes\overline{\mathcal{C}})$ by setting

$$h(\sigma^0\otimes\gamma)=\alpha(\gamma),$$

$$h(\sigma^1\otimes\gamma)=\beta(\gamma),$$

$$h(\sigma^{01}\otimes\gamma)=\lambda(\gamma),$$

where γ is some element of $\overline{\mathcal{C}}$. We claim that $(1+\lambda)\cdot\alpha=\beta$ if and only if h is a homotopy from ϕ_α to ϕ_β. Accordingly, we must prove the equivalence

$$d(\lambda)=\alpha+\lambda\{\alpha\}-\beta\circledcirc(1+\lambda)\Leftrightarrow d(h)=0$$

where d is the differential of $\mathrm{Mor}(B^c(\mathcal{C}),\mathcal{P})$.

Because α and β are Maurer-Cartan elements, and by definition of ∂, the second equality is always satisfied for $\sigma^\varepsilon\otimes\gamma$ with $\varepsilon=0,1$ and $\gamma\in\overline{\mathcal{C}}$. We just need to check this equality on terms $\sigma^{01}\otimes\gamma$ for any $\gamma\in\overline{\mathcal{C}}$:

$$\begin{array}{rcl} d(h)(\sigma^{01}\otimes\gamma) & = & d(h(\sigma^{01}\otimes\gamma))-h(\partial(\sigma^{01}\otimes\gamma)) \\ & = & d(\lambda(\gamma))-\lambda(d(\gamma))-\alpha(\gamma)+\beta(\gamma) \\ & & -\lambda\{\alpha\}_1(\gamma)+(\beta\circledcirc(1+\lambda)(\gamma)-\beta(\gamma)) \\ & = & d(\lambda)(\gamma)-\alpha(\gamma)-\lambda\{\alpha\}_1(\gamma)+\beta\circledcirc(1+\lambda)(\gamma). \end{array}$$

We then have the desired equivalence. □

Chapter 2

Pre-Lie algebras up to homotopy with divided powers and homotopy of operadic mapping spaces

The purpose of this memoir is to study pre-Lie algebras up to homotopy with divided powers, and to use this algebraic structure for the study of mapping spaces in the category of operads. We define a new notion of algebra called $\Gamma\Lambda\mathcal{PL}_\infty$-algebra which characterizes the notion of $\Gamma(\mathcal{PreLie}_\infty, -)$-algebra. We also define a notion of a Maurer-Cartan element in complete $\Gamma\Lambda\mathcal{PL}_\infty$-algebras which generalizes the classical definition in Lie algebras. We prove that for every complete brace algebra A, and for every $n \geq 0$, the tensor product $A \otimes \Sigma N^*(\Delta^n)$ is endowed with the structure of a complete $\Gamma\Lambda\mathcal{PL}_\infty$-algebra, and define the simplicial Maurer-Cartan set $\mathcal{MC}_\bullet(A)$ associated to A as the Maurer-Cartan set of $A \otimes \Sigma N^*(\Delta^\bullet)$. We compute the homotopy groups of this simplicial set, and prove that the functor $\mathcal{MC}_\bullet(-)$ satisfies a homotopy invariance result, which extends the Goldman-Millson theorem in dimension 0. As an application, we give a description of mapping spaces in the category of non-symmetric operads in terms of this simplicial Maurer-Cartan set. We establish a generalization of the latter result for symmetric operads.

Outline of the current chapter

Introduction

The usual category of topological spaces comes equipped with a functor $\mathrm{Map}_{\mathcal{T}op}(-,-) : \mathcal{T}op^{op} \times \mathcal{T}op \longrightarrow \mathrm{sSet}$ which endows $\mathcal{T}op$ with the structure of a simplicial category (see for instance [Fre17b, §2.1.1]). This functor can be used in order to handle higher homotopies in the category $\mathcal{T}op$. For every topological spaces X, Y, the connected components of $\mathrm{Map}_{\mathcal{T}op}(X,Y)$ are in bijection with homotopy classes of morphisms $X \longrightarrow Y$, while the homotopy groups encode higher homotopy relations. This approach allows us to use tools from algebraic topology in order to study homotopy morphisms from X to Y. The functor $\mathrm{Map}_{\mathcal{T}op}(-,-)$ is defined as follows. For every $X \in \mathcal{T}op$, we define two functors $X \otimes - : \mathrm{sSet} \longrightarrow \mathcal{T}op$ and $X^- : \mathrm{sSet}^{op} \longrightarrow \mathcal{T}op$ by

$$X \otimes K := X \times |K| \ ; \ X^K := \mathrm{Mor}_{\mathcal{T}op}(|K|, X),$$

for every $X \in \mathcal{T}op$ and $K \in \mathrm{sSet}$, where $|K| \in \mathcal{T}op$ denotes the geometric realization of the simplicial set K. For every $X, Y \in \mathcal{T}op$ and $K \in \mathrm{sSet}$, we have the isomorphism

$$\mathrm{Mor}_{\mathcal{T}op}(X \otimes K, Y) \simeq \mathrm{Mor}_{\mathcal{T}op}(X, Y^K).$$

We then define $\mathrm{Map}_{\mathcal{T}op}(X, Y)$ as the simplicial set $\mathrm{Mor}_{\mathcal{T}op}(X \otimes \Delta^\bullet, Y)$, where, for every $n \geq 0$, we denote by Δ^n the fundamental n-simplex.

In a general model category C, we have an analogue of the functors $X \otimes -$ and X^-. Such functors are defined by giving the image of Δ^n for every $n \geq 0$. The cosimplicial set $X \otimes \Delta^\bullet$ is called a *cosimplicial frame* associated to X, while $X^{\Delta^\bullet}$ is called a *simplicial frame* associated to X (see for instance [Fre17b, §3.2.2, §3.2.7]). However, we only have a zig-zag of weak-equivalences of simplicial sets between $\mathrm{Mor}_C(X \otimes \Delta^\bullet, Y)$ and $\mathrm{Mor}_C(X, Y^{\Delta^\bullet})$ instead of an isomorphism, provided that X is cofibrant and Y is fibrant. This still allows us to construct a simplicial set $\mathrm{Map}_C(X, Y)$, which is unique up to a zig-zag of weak-equivalences. As in $\mathcal{T}op$, the connected components of $\mathrm{Map}_C(X, Y)$ are in bijection with homotopy classes of morphisms $X \longrightarrow Y$.

In this memoir, we provide an approach in order to study the homotopy of such mapping spaces in the category of non-symmetric operads and in the category of symmetric operads as well, where in both cases we consider operads defined in the category of differential graded $\mathbb{K}$-modules (dg $\mathbb{K}$-modules for short).

We review the known results in characteristic 0 before explaining our results which deal with the positive characteristic context.

State-of-the-art in characteristic 0

A comprehensive study of the homotopy type of a mapping spaces in the category of symmetric operads has already been done in the case $char(\mathbb{K}) = 0$. Let $\mathcal{C}$ be a coaugmented connected cooperad and $\mathcal{P}$ be an augmented connected operad. The computation of the homotopy groups can be deduced from a description of a mapping space $\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P})$ given in [Yal16], in the context of properads. For this purpose, we use an explicit simplicial frame $\mathcal{P}^{\Delta^\bullet}$ associated to the operad $\mathcal{P}$, given by:

$$\mathcal{P}^{\Delta^\bullet} := \mathcal{P} \otimes \Omega^*(\Delta^\bullet),$$

where, for every $n \geq 0$, we denote by $\Omega^*(\Delta^n)$ the Sullivan algebra of de Rham polynomial forms on Δ^n (see for instance [BG76, §2.1]). The n-simplices of $\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P})$ then correspond to elements in $\mathrm{Hom}_{\Sigma\mathrm{Seq}_\mathbb{K}}(\overline{\mathcal{C}}, \overline{\mathcal{P}}) \widehat{\otimes} \Omega^*(\Delta^n)$ which satisfy some equations, where $\widehat{\otimes}$ is the complete tensor product associated to the complete filtered dg modules $\mathrm{Hom}_{\Sigma\mathrm{Seq}_\mathbb{K}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})$ and $\Omega^*(\Delta^n)$. These equations can be written by using the Lie algebra structure on $\mathrm{Hom}_{\mathrm{Seq}_\mathbb{K}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})$ induced by its pre-Lie algebra structure (see for instance [LV12, §6.4.4] for a definition of the pre-Lie product). We recall the definitions in the paragraphs to follow.

Recall that if L is a complete Lie algebra, then a Maurer-Cartan element is an element $\tau \in L_{-1}$ such that

$$d(\tau) + \frac{1}{2}[\tau, \tau] = 0.$$

We denote by $\mathcal{MC}(L)$ the set of Maurer-Cartan elements in L. Note that every $\tau \in \mathcal{MC}(L)$ induces a differential d_τ defined by

$$d_\tau(x) = d(x) + [x, \tau].$$

We let L^τ be the dg $\mathbb{K}$-module L endowed with the differential d_τ. Using that $\Omega^*(\Delta^n)$ is endowed with the structure of a commutative algebra for every $n \geq 0$, the dg $\mathbb{K}$-module $L\widehat{\otimes}\Omega^*(\Delta^n)$ is endowed with the structure of a Lie algebra. We define the *simplicial Maurer-Cartan set* associated to L as

$$\mathcal{MC}_\bullet(L) = \mathcal{MC}(L\widehat{\otimes}\Omega^*(\Delta^\bullet)).$$

From [Yal16, Theorem 3.12], for every coaugmented cooperad $\mathcal{C}$, and for every augmented operad $\mathcal{P}$, we obtain the following description:

$$\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P}) = \mathcal{MC}_\bullet(\mathrm{Hom}_{\Sigma\mathrm{Seq}_\mathbb{K}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})).$$

The computation of the homotopy groups of the simplicial set $\mathrm{Map}_{\Sigma\mathcal{O}p}(B^c(\mathcal{C}), \mathcal{P})$ can be deduced from the general computations of the homotopy groups of $\mathcal{MC}_\bullet(L)$ associated to a given complete Lie algebra L. These computations have been made in [Ber15, Theorem 1.1]. Explicitly, if L is a complete Lie algebra, then, for every $\tau \in \mathcal{MC}(L)$ and $k \geq 0$, we have the isomorphism

$$\pi_{k+1}(\mathcal{MC}_\bullet(L), \tau) \simeq H_k(L^\tau),$$

where $H_0(L^\tau)$ is endowed with the group structure BCH given by the Baker-Campbell-Hausdorff formula.

The computation of the connected components of $\mathrm{Map}_{\Sigma\mathcal{O}p}(B^c(\mathcal{C}), \mathcal{P})$ can be achieved by using the pre-Lie deformation theory developed in [DSV16]. Recall that a pre-Lie algebra is a dg $\mathbb{K}$-module L endowed with a linear morphism $\star : L \otimes L \longrightarrow L$ such that

$$(x \star y) \star z - x \star (y \star z) = (-1)^{|y||z|}((x \star z) \star y - x \star (z \star y)).$$

In particular, any pre-Lie algebra L is endowed with the structure of a Lie algebra with the bracket $[x, y] = x \star y - (-1)^{|x||y|} y \star x$. In [DSV16], the author generalized the Lie deformation theory to the pre-Lie context. Explicitly, a Maurer-Cartan element τ in a pre-Lie algebra L is an element $\tau \in L_{-1}$ such that

$$d(\tau) + \tau \star \tau = 0.$$

The gauge group $(L_0, BCH, 0)$ can also be written in terms of pre-Lie operations. We consider the subset $1 + L_0 \subset \mathbb{K} \oplus L$. Under some convergence hypothesis, we define the circular product $\circledcirc : L \times (1 + L_0) \longrightarrow L$ by

$$x \circledcirc (1+y) = \sum_{n\geq 0} \frac{1}{n!} x\{\underbrace{y, \ldots, y}_{n}\},$$

for every $x \in L$ and $y \in L_0$, where we denote by $-\{-, \ldots, -\}$ the symmetric brace operations associated to L (see [OG08] or [LM05]). We can restrict this product to an operation on $1 + L_0$ defined by

$$(1+x) \circledcirc (1+y) = 1 + y + \sum_{n\geq 0} \frac{1}{n!} x\{\underbrace{y, \ldots, y}_{n}\}$$

for every $x, y \in L_0$. Then the triple $(1 + L_0, \circledcirc, 1)$ is a group isomorphic to the gauge group (see [DSV16, Theorem 2]). The group $(1 + L_0, \circledcirc, 1)$ also acts on $\mathcal{MC}(L)$ via

$$(1+\mu)\cdot\tau = (\tau + \mu \star \tau - d(\tau)) \circledcirc (1+\mu)^{\circledcirc -1}.$$

We define the Deligne groupoid $\mathrm{Deligne}(L)$ as the category with $\mathcal{MC}(L)$ as set of objects, and $(1 + L_0, \circledcirc, 1)$ as hom-sets.

Using a cylinder object associated to $B^c(\mathcal{C})$ (see [Fre17b, Theorem 3.2.14]) and [DSV16, Corollary 2], we obtain a bijection

$$\pi_0 \mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P}) \simeq \pi_0 \mathrm{Deligne}(\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})),$$

where the right hand-side denotes the set of isomorphism classes of $\mathrm{Deligne}(\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}}))$.

Objectives and Results

If $char(\mathbb{K}) > 0$, then the simplicial set $\mathcal{P} \otimes \Omega^*(\Delta^\bullet)$ given in [Yal16] is no longer a simplicial frame associated to $\mathcal{P}$, as the cohomology of $\Omega^*(\Delta^n)$ is not 0 for every $n \geq 0$.

The first description of $\pi_0 \mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P})$ has been generalized to the positive characteristic context in [Ver23] by using a $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra structure on $\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})$. Recall briefly that a $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra is a dg module endowed with operations $-\{-, \ldots, -\}_{r_1,\ldots,r_n}$, defined for every integers $r_1, \ldots, r_n \geq 0$, and which mimic the operations

$$x\{y_1, \ldots, y_n\}_{r_1,\ldots,r_n} = \frac{1}{\prod_i r_i!} x\{\underbrace{y_1, \ldots, y_1}_{r_1}, \ldots, \underbrace{y_n, \ldots, y_n}_{r_n}\}.$$

This notion has been studied in the non-graded context in [Ces18], and generalized to the graded context in [Ver23]. Following the formulas of [DSV16], the pre-Lie deformation theory can be generalized to a deformation theory controlled by $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebras, which is valid over a ring with positive characteristic. For every $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra L, we thus have a notion of Deligne groupoid $\mathrm{Deligne}(L)$ (see [Ver23, Proposition-Definition 2.30]). Using a $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra structure on $\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})$ (see [Ver23, Corollary 2.18]), we retrieve, by [Ver23, Theorem 3.6], a bijection

$$\pi_0\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}),\mathcal{P}) \simeq \pi_0\mathrm{Deligne}(\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}})).$$

In this memoir, we construct an explicit cosimplicial frame associated to $B^c(\mathcal{C})$, in the case where $\mathcal{C}$ is a non-symmetric cooperad. Explicitly, for ever $n \geq 0$, we construct a twisting derivation ∂^n on the operad $\mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1}N_*(\Delta^n))$ such that

$$B^c(\mathcal{C}) \otimes \Delta^\bullet := (\mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1}N_*(\Delta^\bullet)), \partial^\bullet)$$

is a cosimplicial frame associated to $B^c(\mathcal{C})$ where $N_*(\Delta^n)$ is the normalized chain complex of the simplicial set Δ^n. The n-simplices of a mapping space from $B^c(\mathcal{C})$ to $\mathcal{P}$ can then be identified with elements of $\mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}}) \otimes \Sigma N^*(\Delta^n)$ which satisfy some equations. Our purpose is to interpret these equations as Maurer-Cartan equations. Our main ideas are the following. We deal with $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty,-)$-algebra structures, where $\mathcal{P}re\mathcal{L}ie_\infty$ denotes the operad that governs pre-Lie algebras up to homotopy. The key point is that if A is a brace algebra and if N is an algebra over the Barratt-Eccles operad $\mathcal{E}$, then $A \otimes N$ is a $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty,-)$-algebra. Using this result with $A = \mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}})$ (which is a brace algebra by [LV12, Proposition 6.4.2] and [GV95, Proposition 1]) and $N = N^*(\Delta^n)$ (see [BF04]) precisely give the desired equations.

The $\mathcal{P}re\mathcal{L}ie_\infty$-algebras, also called pre-Lie algebras up to homotopy, have been studied in [CL01]. The author characterized the data of a $\mathcal{P}re\mathcal{L}ie_\infty$-algebra structure on L as the data of brace operations which satisfy some identities. We denote these brace operations by $-\{\!\{-,\ldots,-\}\!\}$ in this memoir, and we assume that these operations defined on the suspension ΣL. As for the study of the monad $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$ in [Ces18], we prove that giving a $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty,-)$-algebra structure on L is equivalent to giving weighted brace operations $-\{\!\{-,\ldots,-\}\!\}_{r_1,\ldots,r_n}$ on the suspension ΣL which are similar to the operations

$$x\{\!\{y_1,\ldots,y_n\}\!\}_{r_1,\ldots,r_n} = \frac{1}{\prod_i r_i!} x\{\!\{\underbrace{y_1,\ldots,y_1}_{r_1},\ldots,\underbrace{y_n,\ldots,y_n}_{r_n}\}\!\}.$$

We give an other characterization of such objects that will emphasize a notion of ∞-morphism. For any graded $\mathbb{K}$-module V, we set

$$\Gamma\mathrm{Perm}^c(V) = \bigoplus_{n\geq 0} V \otimes (V^{\otimes n})^{\Sigma_n}.$$

We prove that $\Gamma\mathrm{Perm}^c(V)$ is endowed with a coproduct $\Delta_{\Gamma\mathrm{Perm}}$ which, in some sense, is compatible with the coproduct defined in [CL01, §2.3] on $\mathrm{Perm}^c(V)$. We then define the category $\Gamma\Lambda\mathcal{PL}_\infty$ formed by pairs (V,Q) where V is a graded $\mathbb{K}$-module and Q a coderivation on $\Gamma\mathrm{Perm}^c(V)$ of degree -1 such that $Q^2 = 0$. A morphism in $\Gamma\Lambda\mathcal{PL}_\infty$, also called an ∞-morphism, is a morphism of coalgebras which preserve the coderivations. We prove that L is a $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty,-)$-algebra if and only if $\Sigma L \in \Gamma\Lambda\mathcal{PL}_\infty$. We also define a notion of (complete) filtered $\Gamma\Lambda\mathcal{PL}_\infty$-algebra. The category of complete $\Gamma\Lambda\mathcal{PL}_\infty$-algebras is denoted by $\widehat{\Gamma\Lambda\mathcal{PL}_\infty}$.

Given an object $V \in \widehat{\Gamma\Lambda\mathcal{PL}_\infty}$, a Maurer-Cartan element is a degree 0 element

$x \in V$ such that
$$d(x) + \sum_{n\geq 1} x\{\!\{x\}\!\}_n = 0.$$
We denote by $\mathcal{MC}(V)$ the set formed by these objects. We prove that any ∞-morphism $\phi : V \rightsquigarrow W$ induces a map
$$\mathcal{MC}(\phi) : \mathcal{MC}(V) \longrightarrow \mathcal{MC}(W)$$
so that $\mathcal{MC} : \widehat{\Gamma\Lambda\mathcal{PL}_\infty} \longrightarrow \mathrm{Set}$ is a functor.

The motivation for using $\Gamma(\mathcal{PreLie}_\infty, -)$-algebras is given by the following theorem.

Theorem D. *Let $\mathcal{B}race$ be the operad which governs brace algebras (see [Cha02, Proposition 2]). There exists an operad morphism $\mathcal{PreLie}_\infty \longrightarrow \mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$ which fits in a commutative square*
$$\begin{tikzcd} \mathcal{PreLie}_\infty \arrow[r] \arrow[d] & \mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E} \arrow[d] \\ \mathcal{PreLie} \arrow[r] & \mathcal{B}race \end{tikzcd}.$$
As brace algebras are endowed with the structure of a $\Gamma(\mathcal{PreLie}, -)$-algebra (see [Ver23, Theorem 2.15]), Theorem D implies that every $\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$-algebra L is a $\Gamma(\mathcal{PreLie}_\infty, -)$-algebra, via the composite
$$\Gamma(\mathcal{PreLie}_\infty, L) \longrightarrow \Gamma(\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}, L) \xleftarrow{\simeq} \mathcal{S}(\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}, L) \longrightarrow L.$$

Using that the normalized cochain complex $N^*(X)$ of a simplicial set X admits the structure of an algebra over the Barratt-Eccles operad (see [BF04]) and Theorem D, we define the simplicial Maurer-Cartan set associated to a complete brace algebra A as
$$\mathcal{MC}_\bullet(A) = \mathcal{MC}(A \otimes \Sigma N^*(\Delta^\bullet)).$$
In particular, the 0-vertices are identified with Maurer-Cartan elements in A, when using its underlying $\Gamma(\mathcal{PreLie}, -)$-algebra structure (see [Ver23, Theorem 2.15]). We explicitly compute the connected components and the homotopy groups of $\mathcal{MC}_\bullet(A)$.

Theorem E. *For every complete brace algebra A, the simplicial set $\mathcal{MC}_\bullet(A)$ is a Kan complex. Moreover, we have the following computations for every $\tau \in \mathcal{MC}(A)$.*

— $\pi_0(\mathcal{MC}_\bullet(A)) \simeq \pi_0\mathrm{Deligne}(A)$, *where* $\mathrm{Deligne}(A)$ *denotes the Deligne groupoid associated to the* $\Gamma(\mathcal{PreLie}, -)$*-algebra A (see [Ver23, Proposition-Definition 2.30]);*

— $\pi_1(\mathcal{MC}_\bullet(A), \tau) \simeq \{h \in A_0 \mid d(h) = \tau + h\langle\tau\rangle - \tau \circledcirc (1 + h)\}/\sim_\tau$, *where $\sim_\tau$ is the equivalence relation such that $h \sim_\tau h'$ if and only if there exists $\psi \in A_1$ such that*
$$h - h' = d(\psi) + \psi\langle\tau\rangle + \sum_{p,q\geq 0} \tau\langle\underbrace{h, \ldots, h}_{p}, \psi, \underbrace{h', \ldots, h'}_{q}\rangle.$$

— $\pi_2(\mathcal{MC}_\bullet(A), \tau) \simeq (H_1(A^\tau), *_\tau, 0)$, *where $*_\tau$ is the group structure on $H_1(A^\tau)$ such that*
$$[\mu] *_\tau [\mu'] = [\mu + \mu' + \tau\langle\mu, \mu'\rangle].$$

— $\pi_{n+1}(\mathcal{MC}_\bullet(A),\tau) \simeq H_n(A^\tau)$ *for every* $n \geq 3$.

We have the following homotopy invariance result, which extends the Goldman-Millson theorem in dimension 0.

Theorem F. *Let* $\Theta : A \longrightarrow B$ *be a morphism of complete brace algebras such that* Θ *is a weak equivalence in* $\mathrm{dgMod}_{\mathbb{K}}$. *Then* $\mathcal{MC}_\bullet(\Theta) : \mathcal{MC}_\bullet(A) \longrightarrow \mathcal{MC}_\bullet(B)$ *is a weak equivalence.*

We use this new deformation theory for the study of the homotopy of mapping spaces in the category of non symmetric operads. For every non-symmetric coaugmented cooperad $\mathcal{C}$ such that $\mathcal{C}(0) = 0$, and for every $n \geq 0$, we construct a twisting derivation ∂^n on the operad $\mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1} N_*(\Delta^n))$ such that

$$B^c(\mathcal{C}) \otimes \Delta^\bullet := (\mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1} N_*(\Delta^\bullet)), \partial^\bullet)$$

is a cosimplicial frame associated to $B^c(\mathcal{C})$. This leads to the following theorem.

Theorem G. *Let* $\mathcal{C}$ *be a coaugmented cooperad and* $\mathcal{P}$ *be an augmented operad such that* $\mathcal{C}(0) = \mathcal{P}(0) = 0$ *and* $\mathcal{C}(1) = \mathcal{P}(1) = \mathbb{K}$. *Then we have an isomorphism of simplicial sets*

$$\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}), \mathcal{P}) \simeq \mathcal{MC}_\bullet(\mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})).$$

The computation of the connected components and the homotopy groups of $\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}), \mathcal{P})$ can then be achieved by using Theorem E.

In the symmetric context, the derivation ∂^n constructed above on $\mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1} N_*(\Delta^n))$ does not preserve the action of the symmetric group for every $n \geq 2$. We instead consider a Σ_*-cofibrant replacement of $B^c(\mathcal{C})$ given by the map $B^c(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}}) \xrightarrow{\sim} B^c(\mathcal{C})$, where $\mathbf{Sur}_{\mathbb{K}}$ is the surjection cooperad defined in [BCN23, Theorem A.1]. Using that the action of Σ_n on $\overline{\mathcal{C}}(n) \otimes \overline{\mathbf{Sur}}_{\mathbb{K}}(n)$ is free for every $n \geq 1$, we construct a twisting derivation ∂^n on the operad $\mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1} N_*(\Delta^n))$ such that

$$B^c(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}}) \otimes \Delta^\bullet := (\mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1} N_*(\Delta^\bullet)), \partial^\bullet)$$

is a cosimplicial frame associated to $B^c(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}})$. We deduce the following theorem.

Theorem H. *Let* $\mathcal{C}$ *be a symmetric coaugmented cooperad and* $\mathcal{P}$ *be a symmetric augmented operad such that* $\mathcal{C}(0) = \mathcal{P}(0) = 0$ *and* $\mathcal{C}(1) = \mathcal{P}(1) = \mathbb{K}$. *Then* $\Sigma\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes N_*(\Delta^\bullet), \overline{\mathcal{P}})$ *is endowed with the structure of a* $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$*-algebra such that we have an isomorphism of simplicial sets*

$$\mathrm{Map}^h_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P}) \simeq \mathcal{MC}(\Sigma\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes N_*(\Delta^\bullet), \mathcal{P})),$$

where $\mathrm{Map}^h_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P}) := \mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}}), \mathcal{P})$

Organization of the memoir

In the first part of this memoir, we recall notions that will be useful for explaining our results. In §2.1.1, we explain our conventions on the context of differential graded $\mathbb{K}$-modules (dg $\mathbb{K}$-modules) in which we carry out our constructions. We examine in particular the definition of dg $\mathbb{K}$-modules which are complete with respect to a filtration and which we use in the definition of Maurer-Cartan elements. In §2.1.2, we review our conventions on operads, and recall the precise definition of algebras with divided powers over an operad. In §2.1.3, we recall the definition of the operad that governs brace algebras and its expression in terms of $\mathbb{K}$-modules of planar rooted trees. In §2.1.4, we recall the definition of the Barratt-Eccles and the definition of the action of this operad on the cochain algebra of simplicial sets through an intermediate operad given by an operad of surjections.

In the second part, we study the structure of a $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebra. In §2.2.1, we recall the construction of the operad $\mathcal{P}re\mathcal{L}ie_\infty$ and a characterization of $\mathcal{P}re\mathcal{L}ie_\infty$-algebras in terms of twisting coderivation on cofree Perm-coalgebras. In §2.2.2, we explain the definition of the category $\Gamma\Lambda\mathcal{PL}_\infty$. In §2.2.3, we explain the definition of the weighted brace operations $-\{-,\ldots,-\}_{r_1,\ldots,r_n}$ and of the notion of a Maurer-Cartan element in a complete $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebra. In §2.2.4, we explain the equivalence between $\Gamma\Lambda\mathcal{PL}_\infty$-algebras and $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebras (up to a shift).

The goal of part 3 is to define the morphism $\mathcal{P}re\mathcal{L}ie_\infty \longrightarrow \mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$ and to prove Theorem D. We actually obtain this morphism as a composite $\mathcal{P}re\mathcal{L}ie_\infty \xrightarrow{(1)} \mathcal{B}race \underset{\mathrm{H}}{\otimes} B^c(\Lambda^{-1}\mathcal{B}race^\vee) \xrightarrow{(2)} \mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$ where (2) is induced by a morphism $B^c(\Lambda^{-1}\mathcal{B}race^\vee) \longrightarrow \mathcal{E}$. In §2.3.1, we explain the construction of the latter morphism $B^c(\Lambda^{-1}\mathcal{B}race^\vee) \longrightarrow \mathcal{E}$. Then, from the general bar duality theory of algebras over operads, every $\mathcal{E}$-algebra A comes with a twisting morphism on the free brace coalgebra $\mathcal{B}race^c(\Sigma A)$. In §2.3.2, we make this twisting morphism explicit in the case $A = N^*(\Delta^n)$ for every $n \geq 0$. We will use this description to control the $\mathcal{P}re\mathcal{L}ie_\infty$-algebra structure on $A \otimes N^*(\Delta^n)$ for every $n \geq 0$, when we study the simplicial Maurer-Cartan set associated to brace algebras. In §2.3.3, we explain the definition of (1) to complete our construction of the morphism $\mathcal{P}re\mathcal{L}ie_\infty \longrightarrow \mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$ and the proof of Theorem D.

In the fourth part, we define and study our notion of a simplicial Maurer-Cartan set associated to a complete brace algebra. In §2.4.1, we define this simplicial set and prove that it is a Kan complex. In §2.4.2, we prove Theorem E, which gives a computation of the connected component and the homotopy groups of the simplicial Maurer-Cartan set associated to a complete brace algebra. In §2.4.3, we give an interpretation of the first differentials computed in §2.3.2 by computing the first simplices of the simplicial Maurer-Cartan set associated to a chosed complete brace algebra defined in the context of associated algebras up to homotopy. In §2.4.4, we prove Theorem F, which is an extension of the classical Goldman-Millson theorem for Lie algebras. In §2.4.5, we prove that, in characteristic 0, our simplicial Maurer-Cartan set is related to the simplicial Maurer-Cartan set defined for Lie algebras via a zig-zag of weak-equivalences.

In the fifth part, we show that we can describe a mapping space from the cobar construction of a coaugmented non-symmetric cooperad to an augmented non-symmetric operad as a simplicial Maurer-Cartan set associated to a complete brace algebra. In §2.5.1, we recall the definition of the free operad generated by a sequence in terms of planar rooted trees with inputs, and recall the model structure on the category of operads. In §2.5.2, we construct a cosimplicial frame associated to the cobar construction of a coaugmented cooperad. In §2.5.3, we prove Theorem G, which shows that we can describe a mapping space from the cobar construction of a coaugmented non-symmetric cooperad $\mathcal{C}$ to an augmented non-symmetric operad $\mathcal{P}$ as as the simplicial Maurer-Cartan set associated to the complete brace algebra $\mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})$.

In the last part of this memoir, we show that we can describe a mapping space from the cobar construction of a coaugmented symmetric cooperad to an augmented symmetric operad as a degree-wise Maurer-Cartan set of some $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebras. In §2.6.1, we recall the definition of the free operad generated by a symmetric sequence in terms of planar rooted trees with inputs, and recall the model structure on the category of symmetric connected operads. In §2.6.2, we construct a cosimplicial frame associated to the cobar construction of a symmetric coaugmented cooperad. In §2.6.3, we prove Theorem H, we show that we can describe a mapping space from the cobar construction of a coaugmented cooperad $\mathcal{C}$ to an augmented operad $\mathcal{P}$ as a degree-wise Maurer-Cartan set of some $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebras.

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2.1 Conventions and notations

The goal of this section is to give recollections that will be needed in this memoir, and to set on our notations and conventions.

In §2.1.1, we recall basic definitions on dg $\mathbb{K}$-modules, and give the notation used in this memoir. We also give our definitions and notation on the notion of a (complete) filtered dg $\mathbb{K}$-module, with underlying category $\mathrm{dg}\widehat{\mathrm{Mod}}_{\mathbb{K}}$.

In §2.1.2, we recall the notion of an operad and a cooperad. We also recall the operation of the Hadamard tensor product, which is widely used in this memoir. From the definition, we recall the notion of (co)operadic suspension, and study the (co)algebras over suspensions. We finally recall the notion of a $\Gamma(\mathcal{P}, -)$-algebra associated to an operad $\mathcal{P}$ such that $\mathcal{P}(0) = 0$.

In §2.1.3, we recall the definition of the operad $\mathcal{B}race$ which governs brace algebras in terms of planar rooted trees. We also set on our notations and conventions on planar rooted trees in this subsection.

In §2.1.4, we recall the definition of the Barratt-Eccles and surjection operads, following the conventions of [BF04]. We also recall an important example of an algebra over such operads, given by the normalized cochain complex of simplicial sets.

In §2.1.5, we recall notions and notations on permutations and symmetric groups. More precisely, we recall the notion of shuffle permutations which gives a set of representatives of $\Sigma_m/\Sigma_{r_1} \times \cdots \times \Sigma_{r_n}$ where $r_1 + \cdots + r_n = m$.

2.1.1 The category $\mathrm{dgMod}_{\mathbb{K}}$

Let $\mathbb{K}$ be a field. In this memoir, we work in the category $\widehat{\mathrm{dgMod}}_{\mathbb{K}}$ that we aim to define in this subsection.

A *graded* $\mathbb{K}$*-module* is a $\mathbb{K}$-module V equipped with a decomposition

$$V \simeq \bigoplus_{n\in\mathbb{Z}} V_n.$$

Given such a decomposition, an element $x \in V$ is *homogeneous* if $x \in V_n$ for some $n \in \mathbb{Z}$. The integer n is called the *degree* of x, and denoted by $|x|$. A morphism of graded $\mathbb{K}$-modules of degree d is a morphism $f : V \longrightarrow W$ of $\mathbb{K}$-modules such that $f(V_n) \subset W_{n+d}$. We denote by $\mathrm{Hom}(V,W)_d$ the $\mathbb{K}$-module formed by such morphisms. We set

$$\mathrm{Hom}(V,W) := \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}(V,W)_d.$$

We denote by $\mathrm{gMod}_{\mathbb{K}}$ the category formed by graded $\mathbb{K}$-modules with as set of morphisms from V to W the $\mathbb{K}$-module $\mathrm{Hom}(V,W)_0$. The *dual* graded $\mathbb{K}$-module of V, denoted by $V^\vee$, is defined by $V^\vee = \mathrm{Hom}(V,\mathbb{K})$ where $\mathbb{K}$ is the graded $\mathbb{K}$-module with only one degree 0 component given by $\mathbb{K}$. Explicitly, we have

$$V^\vee \simeq \bigoplus_{n\in\mathbb{Z}} V^\vee_{-n}.$$

If V is finite dimensional, then, given a basis $x_1,\ldots,x_n$ of V, we endow $V^\vee$ with the basis $x_1^\vee,\ldots,x_n^\vee$ where, for every $1 \leq i \leq n$, the linear form $x_i^\vee \in \mathrm{Hom}(V,\mathbb{K})$ is defined by

$$x_i^\vee(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}.$$

A *differential* on V is a degree -1 morphism $d_V : V \longrightarrow V$ such that $d_V \circ d_V = 0$. We usually omit the index V if there is no ambiguity on the ambient $\mathbb{K}$-module. The pair (V, d_V) is called a *differential graded* $\mathbb{K}$*-module* (or dg $\mathbb{K}$-module). A morphism of dg $\mathbb{K}$-modules is a morphism of graded $\mathbb{K}$-modules which commutes with the differentials. If V and W are dg $\mathbb{K}$-modules, then the graded $\mathbb{K}$-module $\mathrm{Hom}(V,W)$ comes equipped with a differential $d = d_{\mathrm{Hom}(V,W)}$ defined by

$$d(f) = d_W \circ f - (-1)^{|f|} f \circ d_V.$$

We denote by $\mathrm{dgMod}_{\mathbb{K}}$ the category formed by dg $\mathbb{K}$-modules with as hom-sets the previous dg $\mathbb{K}$-module. This category is endowed with the structure of a symmetric monoidal category: the tensor product $V \otimes W$ of two elements $V, W \in \mathrm{dgMod}_{\mathbb{K}}$ is the usual tensor product of $\mathbb{K}$-modules, with as degree n component

$$(V \otimes W)_n = \bigoplus_{p+q=n} V_p \otimes W_q.$$

The differential on $V \otimes W$ is defined by

$$d_{V\otimes W}(v \otimes w) = d_V(v) \otimes w + (-1)^{|v|} v \otimes d_W(w).$$

The symmetry operator $\tau : V \otimes W \longrightarrow W \otimes V$ is defined by

$$\tau(v \otimes w) = (-1)^{|v||w|} w \otimes v.$$

The tensor product $f \otimes g$ of two morphisms of dg $\mathbb{K}$-modules $f : V \longrightarrow V'$ and $g : W \longrightarrow W'$ is defined by

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w).$$

Note that, as in the non-graded setting, we have an isomorphism of dg $\mathbb{K}$-modules

$$\mathrm{Hom}(U \otimes V, W) \xrightarrow{\simeq} \mathrm{Hom}(U, \mathrm{Hom}(V, W))$$

for every dg $\mathbb{K}$-modules U, V and W, defined by sending a morphism $f : U \otimes V \longrightarrow W$ to the morphism which sends $u \in U$ to the morphism $v \in V \longmapsto f(u \otimes v) \in W$.

If V is finite dimensional, we also have an isomorphism of dg $\mathbb{K}$-modules

$$\mathrm{Hom}(V, W) \xrightarrow{\simeq} W \otimes V^\vee.$$

This morphism is defined by sending $f \in \mathrm{Hom}(V, W)$ to $\sum_{i=1}^n f(e_i) \otimes e_i^\vee$, where $e_1, \dots, e_n$ is a chosen basis of V. Using the two above isomorphisms gives

$$(V \otimes W)^\vee \simeq W^\vee \otimes V^\vee,$$

provided that V is finite dimensional.

If we set $V = A^{\otimes n}, V' = C^{\otimes k}$ and $W = B^{\otimes n}, W' = D^{\otimes k}$ for some dg $\mathbb{K}$-modules A, B, C, D and $n, k \geq 0$, then $f \otimes g$ is a morphism from $A^{\otimes n} \otimes B^{\otimes n}$ to $C^{\otimes k} \otimes D^{\otimes k}$. For our needs, it will sometimes be more convenient to see $f \otimes g$ as a morphism from $(A \otimes B)^{\otimes n}$ to $(C \otimes D)^k$.

Definition 2.1.1. *Let $f : A^{\otimes n} \longrightarrow C^{\otimes k}$ and $g : B^{\otimes n} \longrightarrow D^{\otimes k}$. We denote by $f \widetilde{\otimes} g : (A \otimes B)^{\otimes n} \longrightarrow (C \otimes D)^{\otimes k}$ the morphism defined by the following commutative diagram:*

$$\begin{tikzcd}
A^{\otimes n} \otimes B^{\otimes n} \arrow[r, "f\otimes g"] & C^{\otimes k} \otimes D^{\otimes k} \\
(A \otimes B)^{\otimes n} \arrow[u, "\simeq"] \arrow[r, "f\widetilde{\otimes} g"'] & (C \otimes D)^{\otimes k} \arrow[u, "\simeq"']
\end{tikzcd}$$

where we consider the isomorphisms given by the symmetry operator.

We now recall the definition of the suspension of dg $\mathbb{K}$-modules.

Definition 2.1.2. *Let $k \in \mathbb{Z}$ and $V \in \mathrm{dgMod}_{\mathbb{K}}$. We denote by Σ^k the dg $\mathbb{K}$-module generated by one degree k element, also denoted by Σ^k, with 0 as differential. We define the k-*suspension *of V as*

$$\Sigma^k V = \Sigma^k \otimes V.$$

For every $v \in V$, we set $\Sigma^k v := \Sigma^k \otimes v$. We also set $\Sigma^1 = \Sigma$.

For every $n, k \geq 0$, we have an isomorphism of $\mathbb{K}$-modules $(\Sigma^k V)_n \simeq V_{n-k}$. Besides, giving a degree k morphism $V \longrightarrow W$ is equivalent to giving a degree 0 morphism $V \longrightarrow \Sigma^k W$, and also equivalent to giving a degree 0 morphism $\Sigma^{-k} V \longrightarrow W$.

Note that, for every $k \in \mathbb{Z}$, the k-suspension defines an endofunctor in the category $\mathrm{dgMod}_{\mathbb{K}}$: for every $f \in \mathrm{Hom}(V, W)$, we define $\Sigma^k f \in \mathrm{Hom}(\Sigma^k V, \Sigma^k W)$ by

$$(\Sigma^k f)(\Sigma^k v) = (-1)^{k|f|} \Sigma^k f(v)$$

for every $v \in V$.

We now make explicit the notion of filtration that we consider in this memoir.

Definition 2.1.3. *Let $V \in \mathrm{dgMod}_{\mathbb{K}}$. A* filtration *on V is a sequence $(F_n V)_{n \geq 1}$ of sub dg $\mathbb{K}$-modules of V such that*

$$\cdots \subset F_n V \subset F_{n-1} V \subset \cdots \subset F_1 V = V.$$

A dg $\mathbb{K}$-module endowed with a filtration is called a filtered *dg $\mathbb{K}$-module. A filtered dg $\mathbb{K}$-module V is said to be* complete *if we have an isomorphism*

$$V \simeq \lim_{\longleftarrow} V / F_n V.$$

For every filtered dg $\mathbb{K}$-module V, the *completion* of V with respect to its filtration is the filtered dg $\mathbb{K}$-module defined by

$$\widehat{V} = \lim_{\longleftarrow} V / F_n V,$$

with as filtration

$$F_m \widehat{V} = \lim_{\longleftarrow} F_m V / (F_n V \cap F_m V).$$

We immediately see that $\widehat{V}$ is complete.

Remark 2.1.4. *If V is complete with respect to the filtration $(F_n V)_{n \geq 1}$, then $\bigcap_{n \geq 1} F_n V = 0$. This implies that if $x \in V$ is such that $x \in F_k V \implies x \in F_{k+1} V$ for every $k \geq 1$, then $x = 0$.*

Let $V, W \in \mathrm{dgMod}_{\mathbb{K}}$ be two complete filtered dg $\mathbb{K}$-modules. We say that a morphism $f : V \longrightarrow W$ *preserves the filtrations* if it satisfies, for all $n \geq 1$,

$$f(F_n V) \subset F_n W.$$

The complete filtered dg $\mathbb{K}$-modules together with the filtration preserving morphisms define a category denoted by $\mathrm{dg}\widehat{\mathrm{Mod}}_{\mathbb{K}}$. If V and W are filtered, then their tensor product $V \otimes W$ is also filtered with

$$F_n(V \otimes W) = \sum_{p+q=n} F_p V \otimes F_q W.$$

However, this filtered dg $\mathbb{K}$-module is not complete in general, even if V and W are so. We therefore define the *complete tensor product* by

$$V \widehat{\otimes} W = \varprojlim (V \otimes W)/F_n(V \otimes W).$$

We can check that the category $\mathrm{dg}\widehat{\mathrm{Mod}}_{\mathbb{K}}$ endowed with $\widehat{\otimes}$ is a symmetric monoidal category.

2.1.2 The notion of an operad and a cooperad

We briefly recall the notion of an operad and its dual notion, the notion of a cooperad. We will mostly follow [Fre17b] and [LV12].

Let $\mathrm{Seq}_{\mathbb{K}}$ be the category whose objects are sequences in $\mathrm{dgMod}_{\mathbb{K}}$. For every $M, N \in \mathrm{Seq}_{\mathbb{K}}$, we denote by $\mathrm{Hom}(M, N)$ the sequence defined for every $n \geq 0$ by

$$\mathrm{Hom}(M, N)(n) = \mathrm{Hom}(M(n), N(n)).$$

Definition 2.1.5. *A* symmetric sequence *is a sequence $M \in \mathrm{Seq}_{\mathbb{K}}$ such that, for every $n \geq 0$, the dg $\mathbb{K}$-module $M(n)$ comes equipped with an action of Σ_n on it. A morphism of symmetric sequences is a morphism of sequences which preserves the actions of the symmetric groups.*

We denote by $\Sigma\mathrm{Seq}_{\mathbb{K}}$ the subcategory of symmetric sequences. Note that if $M, N \in \Sigma\mathrm{Seq}_{\mathbb{K}}$, then $\mathrm{Hom}(M, N) \in \Sigma\mathrm{Seq}_{\mathbb{K}}$ with the action defined by

$$(\sigma \cdot f)(x) = \sigma \cdot f(\sigma^{-1} \cdot x)$$

for every $n \geq 0, f \in \mathrm{Hom}(M(n), N(n)), x \in M(n)$ and $\sigma \in \Sigma_n$.

Definition 2.1.6.

— *An* operad *is a symmetric sequence $\mathcal{P} \in \Sigma\mathrm{Seq}_{\mathbb{K}}$ endowed with composition products*

$$\gamma : \mathcal{P}(n) \otimes \mathcal{P}(r_1) \otimes \cdots \otimes \mathcal{P}(r_n) \longrightarrow \mathcal{P}\left(\sum_i r_i\right),$$

which satisfy associativity, unit and symmetry axioms. The underlying category is denoted by $\Sigma\mathcal{O}p$.

— *Dually, a* cooperad *is a symmetric sequence $\mathcal{C} \in \Sigma\mathrm{Seq}_{\mathbb{K}}$ endowed with composition coproducts*

$$\Delta : \mathcal{C}\left(\sum_i r_i\right) \longrightarrow \mathcal{C}(n) \otimes \mathcal{C}(r_1) \otimes \cdots \otimes \mathcal{C}(r_n),$$

which satisfy coassociativity, counit and symmetry axioms. The underlying category is denoted by $\Sigma\mathcal{O}p^c$.

By forgetting the action of the symmetric groups, and the symmetry axioms, we have the notion of a non-symmetric (co)operad. We denote by $\mathcal{O}p$ and $\mathcal{O}p^c$ the underlying categories.

Remark 2.1.7. *If* $\mathcal{P}$ *is an operad such that* $\mathcal{P}(n)$ *is finite dimensional for every* $n \geq 0$ *and* $\mathcal{P}(0) = 0$*, then the symmetric sequence*

$$\mathcal{P}^\vee(n) := \mathcal{P}(n)^\vee$$

is endowed with the structure of a cooperad given by the dualization of the operadic structure of $\mathcal{P}$.

If $\mathcal{P}$ is an operad, then we define, for every $n, m \geq 0$ and $1 \leq i \leq n$, the *i-th partial composition morphism* by

$$\circ_i : \mathcal{P}(p) \otimes \mathcal{P}(q) \xrightarrow{\simeq} \mathcal{P}(p) \otimes \mathbb{K} \otimes \cdots \otimes \underset{i}{\mathcal{P}(q)} \otimes \cdots \otimes \mathbb{K} \xrightarrow{\gamma} \mathcal{P}(p+q-1)$$

where we plug operadic units in places $j \neq i$. Dually, if $\mathcal{C}$ is a cooperad, we define the *i-th partial decomposition morphism* by

$$\Delta_i : \mathcal{C}(p+q-1) \xrightarrow{\Delta} \mathcal{C}(p) \otimes \mathbb{K} \otimes \cdots \otimes \underset{i}{\mathcal{C}(q)} \otimes \cdots \otimes \mathbb{K} \xrightarrow{\simeq} \mathcal{C}(p) \otimes \mathcal{C}(q)$$

where we plug cooperadic counits in places $j \neq i$.

Example 2.1.8. *For every* $n \geq 0$*, we set* $\mathcal{C}om(n) = \mathbb{K}$ *endowed with the trivial* Σ_n*-action. The isomorphism* $\mathbb{K} \otimes \mathbb{K} \simeq \mathbb{K}$ *endows* $\mathcal{C}om$ *with the structure of an operad called the* commutative operad.

Example 2.1.9. *Let* $V \in \mathrm{dgMod}_{\mathbb{K}}$. *We define the symmetric sequences* End_V *and* CoEnd_V *by*

$$\begin{aligned} \mathrm{End}_V(n) &= \mathrm{Hom}(V^{\otimes n}, V); \\ \mathrm{CoEnd}_V(n) &= \mathrm{Hom}(V, V^{\otimes n}). \end{aligned}$$

These symmetric sequences are endowed with the structure of a symmetric operad defined as follows. Let $f \in \mathrm{End}_V(p), g \in \mathrm{End}_V(q)$ *and* $1 \leq i \leq p$. *We set*

$$f \circ_i g = f \circ (id_V \otimes \cdots \otimes \underset{i}{g} \otimes \cdots \otimes id_V).$$

Let $\phi \in \mathrm{CoEnd}_V(p), \psi \in \mathrm{CoEnd}_V(q)$ *and* $1 \leq i \leq p$. *We set,*

$$\phi \circ_i \psi = (-1)^{|\phi||\psi|}(id_V \otimes \cdots \otimes \underset{i}{\psi} \otimes \cdots \otimes id_V) \circ \phi.$$

These operads are called respectively called the endomorphism *and* coendomorphism *operads generated by* V.

Remark 2.1.10. *Let* $n \geq 0$ *and* $\mathcal{P}$ *be an operad. The elements of* $\mathcal{P}(n)$ *are seen as operations with abstract variables labeled by* $1, \ldots, n$. *In this memoir, we often label* *these variables by elements of a finite set X with n elements. This can be formalized as follows. Let X be a set with n elements, and $\Sigma(n, X)$ be the set of bijections from $[\![1, n]\!]$ to X. We set*

$$\mathcal{P}(X) = (\mathcal{P}(n) \otimes \mathbb{K}[\Sigma(n, X)])_{\Sigma_n},$$

where we make coincide the action of Σ_n on $\mathcal{P}(n)$ with its action by right translation on $\Sigma(n, X)$. The group of permutations on X acts on $\mathcal{P}(X)$ by left translation on $\Sigma(n, X)$. Note that, for every finite sets X, Y with n elements, every bijection $u : X \longrightarrow Y$ induces a morphism $u_* : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$, so that $\mathcal{P}$ defines a functor from the category of finite sets to the category of dg $\mathbb{K}$-modules.

For our needs, we apply the above construction to totally ordered finite sets. In this setting, we can shape operadic compositions on finite sets in the following way. Let $X = x_1 < \cdots < x_n$ and $Y = y_1 < \cdots < y_n$ be two disjoint totally ordered sets. We denote by $x : [\![1, n]\!] \longrightarrow X$ and $y : [\![1, m]\!] \longrightarrow Y$ the unique order preserving maps. Let $1 \leq i \leq n$. We set

$$X \sqcup_i Y = x_1 < \cdots < x_{i-1} < y_1 < \cdots < y_n < x_{i+1} < \cdots < x_n.$$

Let $z : [\![1, n+m-1]\!] \longrightarrow X \sqcup_i Y$ be the order preserving map. We define

$$\circ_i : \mathcal{P}(X) \otimes \mathcal{P}(Y) \longrightarrow \mathcal{P}(X \sqcup_i Y)$$

by the following commutative diagram:

$$\begin{tikzcd} \mathcal{P}(n) \otimes \mathcal{P}(m) \arrow[r, "\circ_i"] \arrow[d, "x_* \otimes y_*"', "\simeq"] & \mathcal{P}(n+m-1) \arrow[d, "\simeq"', "z_*"] \\ \mathcal{P}(X) \otimes \mathcal{P}(Y) \arrow[r, "\circ_i"'] & \mathcal{P}(X \sqcup_i Y) \end{tikzcd} .$$

Definition 2.1.11. *Let $\mathcal{P}$ be an operad. A* $\mathcal{P}$-algebra *A (respectively* $\mathcal{P}$-coalgebra *C) is the data of a dg $\mathbb{K}$-module A (resp. C) together with an operad morphism $\mathcal{P} \longrightarrow \mathrm{End}_A$ (respectively $\mathcal{P} \longrightarrow \mathrm{CoEnd}_C$).*

If A is a $\mathcal{P}$-algebra with associated morphism $\phi : \mathcal{P} \longrightarrow \mathrm{End}_A$, for every $p \in \mathcal{P}(n)$, we set $p^A := \phi(p) \in \mathrm{Hom}(A^{\otimes n}, A)$.
Analogously, if C is a $\mathcal{P}$-coalgebra with associated morphism $\phi : \mathcal{P} \longrightarrow \mathrm{CoEnd}_A$, we set $p^C := \phi(p) \in \mathrm{Hom}(C, C^{\otimes n})$.

Remark 2.1.12. *The above definition is such that the dual dg $\mathbb{K}$-module $C^\vee$ of every $\mathcal{P}$-coalgebra C is endowed with the structure of a $\mathcal{P}$-algebra. We define an operad morphism $\phi^\vee : \mathcal{P} \longrightarrow \mathrm{End}_{C^\vee}$ as follows. Let $\phi : \mathcal{P} \longrightarrow \mathrm{CoEnd}_C$ be the coalgebra structure given by $\mathcal{P}$ on C. For every $n \geq 0$, $p \in \mathcal{P}(n)$ and $u_1, \ldots, u_n \in C^\vee$, we set*

$$\phi^\vee(p)(u_1 \otimes \cdots \otimes u_n) = \phi(p)^\vee(u_1 \otimes \cdots \otimes u_n).$$

In the following, we also use the notion of a complete $\mathcal{P}$-algebra. A *filtered* $\mathcal{P}$-algebra is a filtered dg $\mathbb{K}$-module A endowed with the structure of a $\mathcal{P}$-algebra such that, for every $n \geq 0$ and $p \in \mathcal{P}(n)$, the morphism $p^A : A^{\otimes n} \longrightarrow A$ preserves the filtrations. A *complete* $\mathcal{P}$-algebra is a filtered $\mathcal{P}$-algebra which is complete with respect to its filtration.

We define a monoidal structure on $\Sigma\mathcal{O}p$ and $\Sigma\mathcal{O}p^c$ given by the *Hadamard tensor product.*

Definition 2.1.13. *Let $\mathcal{P}, \mathcal{Q}$ be two operads and $\mathcal{C}, \mathcal{D}$ be two cooperads.*

— *The* Hadamard tensor product *of $\mathcal{P}$ and $\mathcal{Q}$ is the operad $\mathcal{P} \underset{\mathrm{H}}{\otimes} \mathcal{Q}$ defined by*

$$(\mathcal{P} \underset{\mathrm{H}}{\otimes} \mathcal{Q})(n) = \mathcal{P}(n) \otimes \mathcal{Q}(n)$$

and equipped with the tensor-wise operadic composition product and the diagonal action of Σ_n.

— *The* Hadamard tensor product *of $\mathcal{C}$ and $\mathcal{D}$ is the cooperad $\mathcal{C} \underset{H}{\otimes} \mathcal{D}$ defined by*

$$(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathcal{D})(n) = \mathcal{C}(n) \otimes \mathcal{D}(n)$$

and equipped with the tensor-wise cooperadic composition coproduct and the diagonal action of Σ_n.

We now define the notion of suspension in the category of operads and cooperads.

Definition 2.1.14. *Let $\mathcal{P}$ be an operad and $\mathcal{C}$ be a cooperad. We set $\Lambda^k = \mathrm{End}_{\Sigma^k}$ and $\Lambda := \Lambda^1$.*

— *The k* operadic suspension *of $\mathcal{P}$ is the operad $\Lambda^k\mathcal{P}$ defined by*

$$\Lambda^k\mathcal{P} = \Lambda^k \underset{\mathrm{H}}{\otimes} \mathcal{P}.$$

— *The k* cooperadic suspension *of $\mathcal{C}$ is the cooperad $\Lambda^k\mathcal{C}$ defined by*

$$\Lambda^k\mathcal{C} = (\Lambda^k)^\vee \underset{\mathrm{H}}{\otimes} \mathcal{C}.$$

Accordingly, $\Lambda^k\mathcal{P}(n) \simeq \Sigma^{k(1-n)}\mathcal{P}(n)$ and $\Lambda^k\mathcal{C}(n) \simeq \Sigma^{k(1-n)}\mathcal{C}(n)$.

We have an isomorphism of operads $\Lambda^{-k}\Lambda^k\mathcal{P} \longrightarrow \mathcal{P}$ given by $(\Sigma^{-k(1-n)}(\Sigma^{k(1-n)}p)) \longmapsto -(-1)^{\frac{n(n+1)}{2}k}p$ for every $p \in \mathcal{P}(n)$.

Note that, for every $k \in \mathbb{Z}$, the dg $\mathbb{K}$-module Σ^k is a Λ^k-algebra. We thus have the following.

Proposition 2.1.15. *Let $\mathcal{P}$ be an operad and $\mathcal{C}$ be a cooperad. Let V be a dg $\mathbb{K}$-module.*

— *Giving a structure of $\mathcal{P}$-algebra on V is equivalent to giving a structure of $\Lambda^k\mathcal{P}$-algebra on Σ^kV.*

— *Giving a structure of $\mathcal{C}$-coalgebra on V is equivalent to giving a structure of $\Lambda^k\mathcal{C}$-coalgebra on Σ^kV.*

Any operad $\mathcal{P}$ gives a monad $\mathcal{S}(\mathcal{P},-) : \mathrm{dgMod}_{\mathbb{K}} \longrightarrow \mathrm{dgMod}_{\mathbb{K}}$ called the *Schur functor* and defined by

$$\mathcal{S}(\mathcal{P},V) = \bigoplus_{n\geq 0} \mathcal{P}(n) \otimes_{\Sigma_n} V^{\otimes n},$$

where we consider the action of Σ_n on $\mathcal{P}(n)$, and the action of Σ_n on $V^{\otimes n}$ by permutation. The monadic structure is given by the composite

$$\mathcal{S}(\mathcal{P},\mathcal{S}(\mathcal{P},V)) \xrightarrow{\simeq} \mathcal{S}(\mathcal{P}\circ\mathcal{P},V) \xrightarrow{\mathcal{S}(\gamma,V)} \mathcal{S}(\mathcal{P},V),$$

where we denote by $\circ$ the composition of symmetric sequences. Note that the algebras over the monad $\mathcal{S}(\mathcal{P},-)$ are precisely the $\mathcal{P}$-algebras.

If $\mathcal{P}(0)=0$, we also have a monad $\Gamma(\mathcal{P},-) : \mathrm{dgMod}_{\mathbb{K}} \longrightarrow \mathrm{dgMod}_{\mathbb{K}}$ defined by

$$\Gamma(\mathcal{P},V) = \bigoplus_{n\geq 1} \mathcal{P}(n) \otimes^{\Sigma_n} V^{\otimes n}.$$

We refer to [Fre00, §1.1.18] for the description of this monadic structure. We only note that we have a morphism of monads

$$Tr : \mathcal{S}(\mathcal{P},V) \longrightarrow \Gamma(\mathcal{P},V)$$

given by the trace map. This is an isomorphism as soon as $char(\mathbb{K})=0$. It is however no longer an isomorphism in general when $char(\mathbb{K})\neq 0$.

Definition 2.1.16. *Let $\mathcal{P}$ be an operad such that $\mathcal{P}(0)=0$. A* $\mathcal{P}$-algebra with divided powers *is a $\Gamma(\mathcal{P},-)$-algebra.*

Note that every $\mathcal{P}$-algebra with divided powers is in particular a $\mathcal{P}$-algebra through the trace map.

Proposition 2.1.17. *Let $\mathcal{P}$ be an operad such that $\mathcal{P}(0)=0$ and V be a dg $\mathbb{K}$-module. Let $k\in\mathbb{Z}$. Then V is a $\Gamma(\mathcal{P},-)$-algebra if and only if $\Sigma^k V$ is a $\Gamma(\Lambda^k\mathcal{P},-)$-algebra.*

Proof. Let V be a $\Gamma(\mathcal{P},-)$-algebra. We endow $\Sigma^k V$ with the structure of a $\Gamma(\Lambda^k\mathcal{P},-)$-algebra via the composite

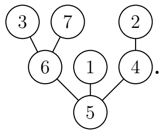
$$\Gamma(\Lambda^k\mathcal{P},\Sigma^k V) \longrightarrow \Sigma^k\Gamma(\mathcal{P},V) \longrightarrow \Sigma^k V,$$

where the first morphism comes from the fact that $\mathrm{End}_{\Sigma^k}(n)\otimes(\Sigma^k)^{\otimes n}$ is isomorphic to Σ^k endowed with the trivial Σ_n action. The fact that this endows $\Sigma^k V$ with a $\Gamma(\Lambda^k\mathcal{P},-)$-algebra structure is an immediate verification. □

2.1.3 On trees and the operad $\mathcal{B}race$

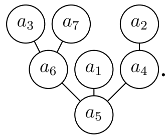
In this section, we recall the notion of a tree and define the operad $\mathcal{B}race$. The notion of a brace algebra was introduced in [GV95, Definition 1], while an explicit construction of their governing operad $\mathcal{B}race$ is given in [Cha02, §2.1-2.2].

Definition 2.1.18. *We call (planar) n-*tree *any simply connected graph endowed with a special vertex called the* root *and a labeling of its set of vertices from* 1 *to n. We put the root at the bottom by convention:*



We denote by $\mathcal{PRT}(n)$ the set of planar rooted trees with n vertices. For every $T \in \mathcal{PRT}(n)$, we set $|T| = n$ and $r(T)$ denotes the root of the tree T.

In some situation, it is more convenient to label an n-tree by a finite set with n elements endowed with a total ordered relation. If X is such a set, we denote by $\mathcal{PRT}(X)$ the set of n-trees labeled with elements of X. Note that since there is a unique order preserving bijection $[\![1, n]\!] \longrightarrow X$, there is a canonical bijection $\mathcal{PRT}(n) \longrightarrow \mathcal{PRT}(X)$. For instance, the tree T shown in the above definition can be seen in $\mathcal{PRT}(a_1 < \cdots < a_7)$ as

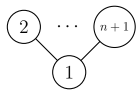


Proposition 2.1.19 ([Cha02, Proposition 2])**.** *Let $\mathcal{B}race$ be the symmetric sequence defined by $\mathcal{B}race = \mathbb{K}[\mathcal{PRT}(n)]$. Then $\mathcal{B}race$ is endowed with the structure of an operad. Its algebras are given by dg $\mathbb{K}$-modules A endowed with morphisms $-\langle -, \ldots, - \rangle : A^{\otimes n+1} \longrightarrow A$ for any $n \geq 0$ such that $x\langle\rangle = x$ and*

$$x\langle y_1, \ldots, y_n\rangle\langle z_1, \ldots, z_p\rangle = \sum \pm x\langle Z_1, y_1\langle Z_2\rangle, \ldots, Z_{2n-1}, y_n\langle Z_{2n}\rangle, Z_{2n+1}\rangle$$

for every $x, y_1, \ldots, y_n, z_1, \ldots, y_p \in A$, where the sum runs over all consecutive subsets such that $Z_1 \sqcup \cdots \sqcup Z_{2n+1} = (z_1, \ldots, z_p)$.

Note that every tree T with $|T| \geq 2$ can be uniquely written as $T = \gamma(F_n(1, T_1, \ldots, T_n))$ where we denote by

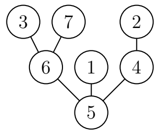
$F_n =$ 

the *corolla* with n leaves.

In the next sections, in order to have formulas which preserve the symmetric groups actions on $\mathcal{B}race$, we pick an explicit set of representatives of $\mathcal{B}race(n)$ as a free Σ_n-set. We achieve this by setting a total order relation on the set of vertices V_T which we call the *canonical order*. For every $a \in \mathbb{N}^*$, we set $V_{a} = a$, and define by induction,

$$V_{\gamma(F_n(a, T_1, \ldots, T_n))} = a < V_{T_1} < \cdots < V_{T_n}$$

for every tree $T_1, \ldots, T_n$. For instance, if we set

$T =$ 

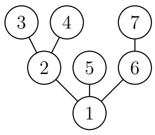
then $V_T = 5 < 6 < 3 < 7 < 1 < 4 < 2$.

Definition 2.1.20. *A tree $T \in \mathcal{PRT}(a_1 < \cdots < a_n)$ is* canonical *(or in the canonical order) if*

$$V_T = a_1 < \cdots < a_n.$$

We let $\sigma_T \in \Sigma_{|T|}$ to be the unique permutation such that $\sigma_T^{-1} \cdot T$ is in the canonical order.

For instance, if we consider the above tree, then $\sigma_T = (5637142)$ and

$\sigma_T^{-1} \cdot T =$ 

is in the canonical order in $\mathcal{PRT}(1 < \cdots < 7)$.

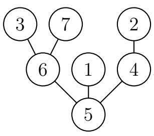
Definition 2.1.21. *Let X be a totally finite ordered set and $T \in \mathcal{PRT}(X)$.*

- *A* subtree *$S \subset T$ of T is an induced simply connected subgraph of T whose set of vertices is seen as a subset Y of X endowed with the induced order relation. Note that $V_S \subset V_T$ as ordered sets.*
- *If $S \subset T$, we define the tree $T/S \in \mathcal{PRT}(X \setminus Y \cup \{S\})$ obtained from T by contracting the tree S on the root of S, denoted by S in the labeling of T/S. The totally ordered set $X \setminus Y \cup \{S\}$ is obtained by changing $r(S)$ into S, and removing all the non-root vertices of S in X.*

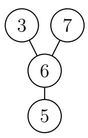
A subtree $S \subset T$ is non-trivial *if neither $|S| \neq 1$ nor $|T/S| \neq 1$.*

Remark 2.1.22. *Let X be a totally finite ordered set. Let $T \in \mathcal{PRT}(X)$ and $S \subset T$. If T is canonical, then so are S and T/S.*

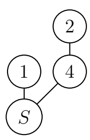
Example: If

$T =$ ,

then

$S =$  $\in \mathcal{PRT}(3 < 5 < 6 < 7)$

is a subtree of T such that

$T/S =$  $\in \mathcal{PRT}(1 < 2 < S < 4)$.

2.1.4 On the Barratt-Eccles and the surjection operads

We devote this subsection to recollections on the Barratt-Eccles operad and the surjection operad. We will mostly follow conventions of [BF04].

Definition 2.1.23. *We let $\mathcal{E}(r)_d$ to be the $\mathbb{K}$-module spanned by $(d+1)$-tuples*

$$(w_0, \dots, w_d) \in (\Sigma_r)^{d+1}$$

with the identification $(w_0, \dots, w_d) \equiv 0$ if $w_i = w_{i+1}$ for some i. We denote by $\mathcal{E}(r)$ the dg $\mathbb{K}$-module with $\mathcal{E}(r)_d$ as degree d component. The differential on $\mathcal{E}(r)$ is defined by

$$d(w_0, \dots, w_d) = \sum_{i=0}^{d} (-1)^i (w_0, \dots, \hat{w}_i, \dots, w_d).$$

We also have an action of Σ_r on $\mathcal{E}(r)$ given by the diagonal action and the left translation of Σ_r on itself.

Proposition 2.1.24. *The symmetric sequence $\mathcal{E}$ is an operad called the* Barratt-Eccles operad.

We refer to [BF04, §1.1] for an explicit description of the composition product. We have an operad morphism $\mathcal{E} \longrightarrow \mathcal{C}om$ obtained by sending each degree 0 element to 1, and sending each non-degree 0 element to 0. This morphism is a weak equivalence arity-wise.

Remark 2.1.25. *The operad $\mathcal{E}$ has the structure of a* Hopf operad. *Namely, we have an operad morphism $\Delta_{\mathcal{E}} : \mathcal{E} \longrightarrow \mathcal{E} \underset{\mathrm{H}}{\otimes} \mathcal{E}$ defined by*

$$\Delta_{\mathcal{E}}(w_0, \dots, w_d) = \sum_{k=0}^{d} (w_0, \dots, w_k) \otimes (w_k, \dots, w_d).$$

We now aim to define the surjection operad χ.

Definition 2.1.26. *Let $r, d \geq 0$. A surjective map $u : [\![1, r+d]\!] \longrightarrow [\![1, r]\!]$ is* degenerate *if $u(i) = u(i+1)$ for some $i \in [\![1, r+d-1]\!]$. We let $\chi(r)_d$ to be the $\mathbb{K}$-module spanned by non-degenerate surjective maps from $[\![1, r+d]\!]$ to $[\![1, r]\!]$.*

In practice, we represent a surjection $u : [\![1, r+d]\!] \longrightarrow [\![1, r]\!]$ by a sequence of values:

$$(u(1) \cdots u(r+d)).$$

Definition 2.1.27. *Let $u \in \chi(r)_d$. An integer $k \in [\![1, r+d]\!]$ is called a* caesura *if $u(k)$ does not represent the last occurrence of its value in u.*

We sometimes represent a surjection by its *table arrangement*, which is defined as follows. Let $u \in \chi(r)_d$. We cut u at the caesuras, in the sense that we set

$$u = (u_0(1) \cdots u_0(r_0)) \cdots (u_d(1) \cdots u_d(r_d)),$$

where $\sum_i r_i = r+d$, and where $u_0(r_0),\dots,u_{d-1}(r_{d-1})$ represent caesuras of u. We then write u as

$$u = \left| \begin{matrix} u_0(1) & \cdots & u_0(r_0) \\ \vdots & & \vdots \\ u_d(1) & \cdots & u_d(r_d) \end{matrix} \right. .$$

We have an obvious action of Σ_r on $\chi(r)_d$ given by the pre-composition.

Proposition 2.1.28 (see [BF04, §1.2])**.** *The symmetric sequence χ is endowed with the structure of a symmetric operad and is called the* surjection operad.

In fact, the surjection operad χ is a quotient of the Barratt-Eccles operad $\mathcal{E}$. The quotient map is called the *table reduction morphism.*

Proposition 2.1.29. *There exists an operad morphism $TR : \mathcal{E} \longrightarrow \chi$ called the table reduction morphism which is surjective arity-wise.*

We refer to [BF04] for more details on the morphism TR. We only recall its definition. Let $w = (w_0,\dots,w_d) \in \mathcal{E}(r)_d$. We set

$$TR(w) = \sum_{r_0+\cdots+r_d=r+d} \left| \begin{matrix} w'_0(1) & \cdots & w'_0(r_0-1) & w'_0(r_0) \\ \vdots & & \vdots & \vdots \\ w'_d(1) & \cdots & w'_d(r_d-1) & w'_d(r_d) \end{matrix} \right.$$

where each row $w'_i(1)\cdots w'_i(r_i)$ represents the first r_i integers occurring in the permutation w_i such that the values $w'_i(1)\cdots w'_i(r_i-1)$ do not occur in

$$\left| \begin{matrix} w'_0(1) & \cdots & w'_0(r_0-1) \\ \vdots & & \vdots \\ w'_{i-1}(1) & \cdots & w'_{i-1}(r_i-1) \end{matrix} \right. .$$

An important example of χ-algebra is given by the normalized cochain complex associated to a simplicial set.

Definition 2.1.30. *Let X be a simplicial set and, for every $k \geq 0$, let $C_k(X)$ be the $\mathbb{K}$-module spanned by X_k. We define a differential on $C_*(X)$ by setting, for every $x \in X_k$,*

$$d(x) = \sum_{i=0}^{k} (-1)^i d_i(x),$$

where we denote by $d_0,\dots,d_k : X_k \longrightarrow X_{k-1}$ the face maps. We then set

$$N_k(X) = C_k(X) / \left(\sum_{i=0}^{k} s_i C_{k-1}(X) \right),$$

where we denote by $s_0,\dots,s_{k-2} : X_{k-1} \longrightarrow X_k$ the degeneracy maps. The dg $\mathbb{K}$-module $N_*(X)$ is called the normalized chain complex of X. *Its dual dg $\mathbb{K}$-module, denoted by $N^*(X)$, is called the* normalized cochain complex of X.

Note that N_* and N^* are functors from sSet to $\mathrm{dgMod}_{\mathbb{K}}$.

Theorem 2.1.31 ([BF04, §2]). *Let $X \in \mathrm{sSet}$. Then $N_*(X)$ is a χ-coalgebra, given by the interval cut operations, which is natural in X. As a consequence, the dg $\mathbb{K}$-module $N^*(X)$ is endowed with the structure of a χ-algebra.*

We refer to [BF04, §2.2.1, §2.2.4] for an explicit description of the interval cut operations. In particular, for every simplicial set X, the dg $\mathbb{K}$-module $N^*(X)$ is endowed with the structure of a $\mathcal{E}$-algebra through the table reduction morphism $TR : \mathcal{E} \longrightarrow \chi$. In this memoir, we mostly consider the case $X = \Delta^n$ for some $n \geq 0$. The elements of $N_d(\Delta^n)$ are linear combination of non-decreasing sequences $a_0 < \cdots < a_d$ of integers in $[\![1, n]\!]$, which we denote by $\underline{a_0 \cdots a_d}$. The normalized chain complex of Δ^n has the following fundamental property.

Proposition 2.1.32. *Let $n \geq 0$ and $0 \leq k \leq n$. Then there exists a deformation retract*

$$\begin{tikzcd} N_*(\Delta^n) \arrow[loop left, "h_n^k"] \arrow[r, shift left, "p_n"] & N_*(\Delta^0), \arrow[l, shift left, "i_n^k"] \end{tikzcd}$$

where $i_n^k : N_*(\Delta^0) \longrightarrow N_*(\Delta^n)$ is the morphism which sends $\underline{0}$ to $\underline{k}$, and $p_n : N_*(\Delta^n) \longrightarrow N_*(\Delta^0)$ is the morphism which sends every vertex to $\underline{0}$.

The claim is that we have the identities

$$\begin{aligned} p_n i_n^k &= id_{N_*(\Delta^0)}; \\ id_{N_*(\Delta^n)} - i_n^k p_n &= dh_n^k + h_n^k d. \end{aligned}$$

We set $\varphi_n^k = i_n^k p_n$. The homotopy h_n^k can be explicitly defined as follows. Let $\underline{a_0 \cdots a_r} \in N_r(\Delta^n)$ be a non-zero element. If this sequence contains k, then we set $h_n^k(\underline{a_0 \cdots a_r}) = 0$. Otherwise we set

$$h_n^k(\underline{a_0 \cdots a_r}) = (-1)^i \underline{a_0 \cdots \overset{i}{k} \cdots a_r},$$

where i is the unique possible position to insert k in $\underline{a_0 \cdots a_r}$ so that we have a non decreasing sequence of integers.

By taking linear duals, we have a similar deformation retract on $N^*(\Delta^n)$. We will keep the same notation $h_n^k : N^*(\Delta^n) \longrightarrow N^{*-1}(\Delta^n)$ and $\varphi_n^k : N^*(\Delta^n) \longrightarrow N^*(\Delta^n)$ for the linear duals of $h_n^k : N_*(\Delta^n) \longrightarrow N_{*+1}(\Delta^n)$ and $\varphi_n^k : N_*(\Delta^n) \longrightarrow N_*(\Delta^n)$.

The dg $\mathbb{K}$-module $I = N^*(\Delta^1)$ can be used to model intervals. We indeed have a decomposition of the diagonal map $\Delta : \mathbb{K} \longrightarrow \mathbb{K}^2$ as

$$\begin{tikzcd} \mathbb{K} = N^*(\Delta^0) \arrow[r, tail, "\sim", "s_0"'] \arrow[rr, bend left, "\Delta"] & N^*(\Delta^1) \arrow[r, two heads, "(d_0, d_1)"'] & N^*(\Delta^0) \times N^*(\Delta^0) = \mathbb{K}^2 \end{tikzcd}$$

where $s_0 = (p_1)^\vee$ and $d_0 = (i_1^0)^\vee$, $d_1 = (i_1^1)^\vee$. We can lift such a diagram in the category of $\mathcal{P} \underset{\mathrm{H}}{\otimes} \mathcal{E}$-algebras for any operad $\mathcal{P}$ to get a construction of a path-object. Recall that a path objet for a $\mathcal{P} \underset{\mathrm{H}}{\otimes} \mathcal{E}$-algebra R is a $\mathcal{P} \underset{\mathrm{H}}{\otimes} \mathcal{E}$-algebra R^I such that the diagonal map

$\Delta : R \longrightarrow R \times R$ can be described as a composite

$$\begin{tikzcd} R \arrow[r, tail, "\sim"', "s_0"] \arrow[rr, bend left, "\Delta"] & R^I \arrow[r, two heads, "(d_0,d_1)"'] & R \times R \end{tikzcd} .$$

Proposition 2.1.33 (see [BF04, §3.1.4, §3.1.9])**.** *Let $\mathcal{P}$ be an operad, and R be a $\mathcal{P} \underset{\mathrm{H}}{\otimes} \mathcal{E}$-algebra. Then*

$$R^I = R \otimes N^*(\Delta^1)$$

is a path objet in the category of $\mathcal{P} \underset{\mathrm{H}}{\otimes} \mathcal{E}$-algebras. The $\mathcal{P} \underset{\mathrm{H}}{\otimes} \mathcal{E}$-algebra structure on R^I is given by the composite

$$\mathcal{P} \underset{\mathrm{H}}{\otimes} \mathcal{E} \xrightarrow{id \otimes \Delta_{\mathcal{E}}} \mathcal{P} \underset{\mathrm{H}}{\otimes} \mathcal{E} \underset{\mathrm{H}}{\otimes} \mathcal{E} \longrightarrow \mathrm{End}_{R \otimes N^*(\Delta^1)} \ ,$$

where we use the $\mathcal{P} \underset{\mathrm{H}}{\otimes} \mathcal{E}$-algebra structure on R, and the $\mathcal{E}$-algebra structure on $N^*(\Delta^1)$.

2.1.5 Appendix: basic results on permutations

In this appendix, we recall basic definitions and notations on permutations and the symmetric groups. Our conventions will follow those given in [Fre17a, §1.1.7]. Let $n \geq 0$. We denote by Σ_n the symmetric group on the elements $1, \ldots, n$. For every $m, n \geq 0$, we denote by $[\![m, n]\!]$ the set of integers k such that $m \leq k \leq n$. We denote by *id* the relevant identity permutation, and we write any permutation $\sigma \in \Sigma_n$ as its sequence of values $(\sigma(1) \cdots \sigma(n))$.

For every $p, q \in \mathbb{N}$ and $\sigma \in \Sigma_p, \tau \in \Sigma_q$, we let $\sigma \oplus \tau \in \Sigma_{p+q}$ to be the permutation defined, for every $1 \leq i \leq p + q$, by

$$(\sigma \oplus \tau)(i) = \begin{cases} \sigma(i) & \text{if } 1 \leq i \leq p \\ \tau(p+i) & \text{if } p+1 \leq i \leq p+q \end{cases} .$$

The operation $\oplus$ is associative in $\bigsqcup_{n \geq 0} \Sigma_n$, so that we can generalize the definition of $\oplus$ to a direct sum of $k \geq 1$ permutations $\sigma_1 \oplus \cdots \oplus \sigma_k$.

Let $r_1, \ldots, r_n \geq 0$ and $\sigma \in \Sigma_n$. We set $\mathbf{r}_i = r_1 + \cdots + r_{i-1} + 1 < \cdots < r_1 + \cdots + r_{i-1} + r_i$. We define the *block permutation* induced by σ of type $(r_1, \ldots, r_r)$ by

$$\sigma_*(r_1, \ldots, r_n) = \mathbf{r}_{\sigma(1)} \cdots \mathbf{r}_{\sigma(n)}.$$

Lemma 2.1.34 ([Fre17a, Proposition 1.1.8])**.** *Let $\sigma \in \Sigma_n$ and $\tau_1 \in \Sigma_{r_1}, \ldots, \tau_n \in \Sigma_{r_n}$. Then*

$$(\tau_1 \oplus \cdots \oplus \tau_n) \cdot \sigma_*(r_1, \ldots, r_n) = \sigma_*(r_1, \ldots, r_n) \cdot (\tau_{\sigma(1)} \oplus \cdots \oplus \tau_{\sigma(n)}).$$

Let $\sigma \in \Sigma_n$ and $\tau_1 \in \Sigma_{r_1}, \ldots, \tau_n \in \Sigma_{r_n}$. We define the permutation $\sigma(\tau_1, \ldots, \tau_n) \in \Sigma_{r_1 + \cdots + r_n}$ by

$$\sigma(\tau_1, \ldots, \tau_n) = (\tau_1 \oplus \cdots \oplus \tau_n) \cdot \sigma_*(r_1, \ldots, r_n).$$

In operads theory, one needs a set of representatives of the quotient $\Sigma_m/\Sigma_{r_1}\times\cdots\times\Sigma_{r_n}$ for every $r_1,\ldots,r_n\geq 0$ such that $r_1+\cdots+r_n=m$. This leads us to the notion of *shuffle permutation.* A $(r_1,\ldots,r_n)$-shuffle permutation is a permutation in $\Sigma_{r_1+\cdots+r_n}$ which preserves the order on each block $\mathbf{r}_1,\ldots,\mathbf{r}_n$. We denote by $Sh(r_1,\ldots,r_n)$ the set of such permutations.
A shuffle permutation $\omega\in Sh(r_1,\ldots,r_n)$ is *pointed* if it satisfies $\omega(1)<\omega(r_1+1)<\cdots<\omega(r_1+\cdots+r_{n-1}+1)$. We denote by $Sh_*(r_1,\ldots,r_n)$ the set of such permutations.

The following results allow us to write any permutations in terms of a product of a shuffle permutation with a composite of a direct sum and a block permutation.

Proposition 2.1.35. *Let $n\geq 0$ and $r_1,\ldots,r_n\geq 0$.*

— *Every $\sigma\in\Sigma_{r_1+\cdots+r_n}$ admits a unique decomposition of the form*

$$\sigma=\omega\cdot(\tau_1\oplus\cdots\oplus\tau_n)$$

where $\tau_i\in\Sigma_{r_i}$ and $\omega\in Sh(r_1,\ldots,r_n)$.

— *Every $\sigma\in\Sigma_{r_1+\cdots+r_n}$ admits a unique decomposition of the form*

$$\sigma=\omega\cdot\sigma(\tau_1,\ldots,\tau_n)$$

where $\tau_i\in\Sigma_{r_i},\sigma\in\Sigma_n$ and $\omega\in Sh_*(r_1,\ldots,r_n)$.

2.2 On $\mathcal{P}re\mathcal{L}ie_\infty$-algebras with divided powers

In this section, we study the structure of $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty,-)$-algebras. The operad $\mathcal{P}re\mathcal{L}ie_\infty$ and its algebras have been explicitly described in [CL01], using the computation of the Koszul dual operad of $\mathcal{P}re\mathcal{L}ie$ given in [Cha01].

In §2.2.1, we recall this explicit construction of $\mathcal{P}re\mathcal{L}ie_\infty$. We also focus on the characterization of the structure of a $\mathcal{P}re\mathcal{L}ie_\infty$-algebra as an algebraic structure on the suspension which we call a $\Lambda\mathcal{PL}_\infty$-algebra.

In §2.2.2, we define the notion of a $\Gamma\Lambda\mathcal{PL}_\infty$-algebra which will be the analogue, in the divided power framework, of a $\Lambda\mathcal{PL}_\infty$-algebra, and we define a notion of ∞-morphism of $\Gamma\Lambda\mathcal{PL}_\infty$-algebras.

In §2.2.3, we define the symmetric weighted braces associated to a $\Gamma\Lambda\mathcal{PL}_\infty$-algebra and the notion of a Maurer-Cartan element in the complete framework. We also prove that giving the structure of a $\Gamma\Lambda\mathcal{PL}_\infty$-algebra is equivalent to giving symmetric brace operations.

In §2.2.4, we prove that giving a structure of a $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty,-)$-algebra is equivalent to giving a structure of a $\Gamma\Lambda\mathcal{PL}_\infty$-algebra on the suspension.

2.2.1 Recollections on pre-Lie algebras up to homotopy

We begin this section by some recollections on the operad Perm, which was introduced by Chapoton in [Cha01]. Let $\mathrm{Perm}(n) = \mathbb{K}^n$. We denote by $(e_i^n)_{1\leq i\leq n}$ the canonical basis of $\mathrm{Perm}(n)$. The group Σ_n acts on $\mathrm{Perm}(n)$ by

$$\sigma \cdot e_i^n = e^n_{\sigma^{-1}(i)}.$$

Proposition 2.2.1. *The symmetric sequence* Perm *is an operad with as compositions*

$$e_i^n(e_{j_1}^{n_1}, \dots, e_{j_k}^{n_k}) = e^{n_1+\cdots+n_k}_{n_1+\cdots+n_{i-1}+j_i}.$$

Theorem 2.2.2 (see [Cha01])**.** *The operad* $\mathcal{P}re\mathcal{L}ie$ *is Koszul and its Koszul dual operad is* $\mathcal{P}re\mathcal{L}ie^! = \mathrm{Perm}$.

This theorem implies that the operad $\mathcal{P}re\mathcal{L}ie_\infty = B^c(\Lambda^{-1}\mathrm{Perm}^\vee)$ gives a model for $\mathcal{P}re\mathcal{L}ie$-algebras up to homotopy. Such algebras have been described by Chapoton and Livernet in [CL01]. We recall this description in the following paragraphs. Let $V \in \mathrm{gMod}_\mathbb{K}$. We set

$$\mathcal{S}(V) := \bigoplus_{n\geq 0} (V^{\otimes n})_{\Sigma_n},$$

where we consider the usual action of Σ_n on $V^{\otimes n}$ by permutation. Note that $\mathcal{S}(V) \simeq \mathbb{K} \oplus \overline{\mathcal{S}}(V)$ with

$$\overline{\mathcal{S}}(V) := \bigoplus_{n\geq 1} (V^{\otimes n})_{\Sigma_n}.$$

Definition 2.2.3 (see [CL01, §2.3])**.** *The* free Perm-coalgebra generated by V *is the graded* $\mathbb{K}$*-module* $\mathrm{Perm}^c(V) = V \otimes \mathcal{S}(V)$ *endowed with the following comultiplication:*
$\Delta_{\mathrm{Perm}}(v_0 \otimes 1) = 0$;

$$\Delta_{\mathrm{Perm}}(v_0 \otimes v_1 \cdots v_n) = \sum_{\substack{0\leq k\leq n-1\\ \sigma\in Sh(k,1,n-k-1)}} \pm (v_0 \otimes v_{\sigma(1)} \cdots v_{\sigma(k)}) \otimes (v_{\sigma(k+1)} \otimes v_{\sigma(k+2)} \cdots v_{\sigma(n)})$$

for every $v_0, \dots, v_n \in V$*, where the sign in the sum is produced by permutations of the factors.*

The coproduct Δ_{Perm} satisfies the following identities (see [CL01, §2.3]):

$$(id \otimes \Delta_{\mathrm{Perm}})\Delta_{\mathrm{Perm}} = (\Delta_{\mathrm{Perm}} \otimes id)\Delta_{\mathrm{Perm}};$$

$$(id \otimes \Delta_{\mathrm{Perm}})\Delta_{\mathrm{Perm}} = (id \otimes \tau)(id \otimes \Delta_{\mathrm{Perm}})\Delta_{\mathrm{Perm}}.$$

Remark 2.2.4. *Let* $\Delta_{\mathcal{S}(V)} : \mathcal{S}(V) \longrightarrow \mathcal{S}(V) \otimes \mathcal{S}(V)$ *be the coproduct defined by* $\Delta_{\mathcal{S}(V)}(1) = 1 \otimes 1$ *and*

$$\Delta_{\mathcal{S}(V)}(v_1 \cdots v_n) = \sum_{k=0}^{n} \sum_{\sigma\in Sh(k,n-k)} \pm (v_{\sigma(1)} \cdots v_{\sigma(k)}) \otimes (v_{\sigma(k+1)} \cdots v_{\sigma(n)})$$

for every $v_1, \ldots, v_n \in V$*. Then* Δ_{Perm} *is given by the composite*

$$\begin{array}{ccc} V \otimes \mathcal{S}(V) \xrightarrow{id \otimes \Delta_{\mathcal{S}(V)}} V \otimes \mathcal{S}(V) \otimes \mathcal{S}(V) & & \\ \downarrow & & \\ V \otimes \mathcal{S}(V) \otimes \overline{\mathcal{S}}(V) \xrightarrow{id \otimes id \otimes i_V} (V \otimes \mathcal{S}(V)) \otimes (V \otimes \mathcal{S}(V)) & & \end{array}$$

where $i_V : \overline{\mathcal{S}}(V) \longrightarrow V \otimes \mathcal{S}(V)$ *is defined by*

$$v_1 \cdots v_n \longmapsto \sum_{k=1}^{n} \pm v_k \otimes v_1 \cdots \widehat{v_k} \cdots v_n$$

for every $v_1, \ldots, v_n \in V$*.*

Let $\pi_V : \mathrm{Perm}^c(V) \longrightarrow V$ be the projection on the first factor.

Proposition 2.2.5 ([CL01, §2.4])**.** *The map*

$$\begin{array}{rcl} \mathit{Coder}(\mathrm{Perm}^c(V)) & \longrightarrow & \mathrm{Hom}(\mathrm{Perm}^c(V), V) \\ d & \longmapsto & \pi_V \circ d \end{array}$$

is a bijection.

Proof. We only recall the construction of the inverse bijection Ψ.
Let $l \in \mathrm{Hom}(\mathrm{Perm}^c(V), V)$. We define $\Psi(l) \in \mathit{Coder}(\mathrm{Perm}^c(V))$ as the sum of the composite

$$\Psi_1(l) : V \otimes \mathcal{S}(V) \xrightarrow{id \otimes \Delta_{\mathcal{S}(V)}} V \otimes \mathcal{S}(V) \otimes \mathcal{S}(V) \xrightarrow{l \otimes id} V \otimes \mathcal{S}(V)$$

and of the composite

$$\begin{array}{l} \Psi_2(l) : V \otimes \mathcal{S}(V) \xrightarrow{\Delta_{\mathrm{Perm}}} (V \otimes \mathcal{S}(V)) \otimes (V \otimes \mathcal{S}(V)) \\ \qquad\qquad\qquad\qquad \downarrow{\scriptstyle id \otimes l} \\ \qquad\qquad\qquad V \otimes \mathcal{S}(V) \otimes V \longrightarrow\!\!\!\!\rightarrow V \otimes \mathcal{S}(V) \end{array}$$

where the last morphism is given by the projection from $\mathcal{S}(V) \otimes V$ to $\mathcal{S}(V)$. One can check that we retrieve the definition given in the proof of [CL01, §2.4]. □

Proposition 2.2.6 ([CL01, §2.5])**.** *Let* $L \in \mathrm{gMod}_{\mathbb{K}}$*. Giving a structure of pre-Lie algebra up to homotopy on* L *is equivalent to giving a degree* -1 *morphism* $l \in \mathrm{Hom}(\mathrm{Perm}^c(\Sigma L), \Sigma L)$ *such that, for every* $x, y_1, \ldots, y_n \in \Sigma L$*, we have*

$$\sum_{i=0}^{n} \sum_{\sigma \in Sh(i,n-i)} \pm l(l(x \otimes y_{\sigma(1)} \cdots y_{\sigma(i)}) \otimes y_{\sigma(i+1)} \cdots y_{\sigma(n)}) \\ + \sum_{i=0}^{n} \sum_{\sigma \in Sh(1,i,n-i-1)} \pm l(x \otimes l(y_{\sigma(1)} \otimes y_{\sigma(2)} \cdots y_{\sigma(i+1)}) \cdot y_{\sigma(i+2)} \cdots y_{\sigma(n)}) = 0$$

where the signs $\pm$ *are produced by the permutations of the elements* $y_1, \ldots, y_n$*.*

In particular, if L is a $\mathcal{P}re\mathcal{L}ie$-algebra up to homotopy, then ΣL is endowed with a differential d given by the restriction $l : \Sigma L \longrightarrow \Sigma L$.

In the following, we adopt the following notation:

$$x\{\!\{y_1, \dots, y_n\}\!\} := l(x \otimes y_1 \cdots y_n).$$

We call such operations the *symmetric braces* associated to the $\mathcal{P}re\mathcal{L}ie_\infty$-algebra L.

Remark 2.2.7. *We have an operad morphism $\mathcal{P}re\mathcal{L}ie_\infty \longrightarrow \mathcal{P}re\mathcal{L}ie$ which sends e_1^2 to the pre-Lie product, and the other e_1^n's to 0. Thus, every $\mathcal{P}re\mathcal{L}ie$-algebra has a canonical $\mathcal{P}re\mathcal{L}ie_\infty$-algebra structure. Beware that the symmetric braces in pre-Lie algebras have nothing to do with the symmetric braces in $\mathcal{P}re\mathcal{L}ie_\infty$-algebras.*

Equivalently, Proposition 2.2.6 asserts that giving a structure of a $\mathcal{P}re\mathcal{L}ie_\infty$-algebra on L is equivalent to giving a coderivation $Q \in Coder(\mathrm{Perm}^c(\Sigma L))$ such that $Q^2 = 0$.

Definition 2.2.8. *We define the category $\Lambda\mathcal{PL}_\infty$ with as set of objects the pairs (V, Q), where $V \in \mathrm{gMod}_{\mathbb{K}}$ and $Q \in Coder(\mathrm{Perm}^c(V))$ is a degree -1 element such that $Q^2 = 0$. A morphism $\phi : (V, Q) \longrightarrow (V', Q')$ in $\Lambda\mathcal{PL}_\infty$ is a morphism of coalgebras $\phi : \mathrm{Perm}^c(V) \longrightarrow \mathrm{Perm}^c(V')$ which preserves the coderivations Q and Q'.*

Usually, a morphism in $\Lambda\mathcal{PL}_\infty$ from (V, Q) to (V', Q') is denoted by $\phi : V \rightsquigarrow V'$ and is called an ∞-*morphism*.

Theorem 2.2.9. *A dg $\mathbb{K}$-module L is a $\mathcal{P}re\mathcal{L}ie_\infty$-algebra if and only if $\Sigma L \in \Lambda\mathcal{PL}_\infty$. Moreover, any morphism of $\mathcal{P}re\mathcal{L}ie_\infty$-algebras $\phi : L \longrightarrow L'$ gives rise to a morphism $\Sigma\phi : \Sigma L \longrightarrow \Sigma L'$ in $\Lambda\mathcal{PL}_\infty$ which preserves the symmetric braces.*

2.2.2 The category $\Gamma\Lambda\mathcal{PL}_\infty$

In this subsection, we aim to define an analogue of the category $\Lambda\mathcal{PL}_\infty$, denoted by $\Gamma\Lambda\mathcal{PL}_\infty$, which will characterize the $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebras as in Theorem 2.2.9.

Let V be a graded $\mathbb{K}$-module. We define $\Gamma(V)$ by

$$\Gamma(V) := \bigoplus_{n \geq 0} (V^{\otimes n})^{\Sigma_n}.$$

We have $\Gamma(V) \simeq \mathbb{K} \oplus \overline{\Gamma}(V)$ with

$$\overline{\Gamma}(V) := \bigoplus_{n \geq 1} (V^{\otimes n})^{\Sigma_n}.$$

Note that we have a morphism $Tr : \mathcal{S}(V) \longrightarrow \Gamma(V)$ called the *trace map* and defined by $Tr(1) = 1$ and

$$Tr(v_1 \cdots v_n) = \sum_{\sigma \in \Sigma_n} \pm v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

for every $v_1, \dots, v_n \in V$ and $n \geq 1$.

Definition 2.2.10. *For every $V \in \mathrm{gMod}_{\mathbb{K}}$, we set*

$$\Gamma\mathrm{Perm}^c(V) := V \otimes \Gamma(V).$$

Our goal is to construct a coproduct $\Delta_{\Gamma\mathrm{Perm}} : \Gamma\mathrm{Perm}^c(V) \longrightarrow \Gamma\mathrm{Perm}^c(V) \otimes \Gamma\mathrm{Perm}^c(V)$ which is compatible, in some sense, with the coproduct $\Delta_{\mathrm{Perm}} : \mathrm{Perm}^c(V) \longrightarrow \mathrm{Perm}^c(V) \otimes \mathrm{Perm}^c(V)$. For this purpose, we first consider the *tensor algebra* generated by V:

$$T(V) := \bigoplus_{n \geq 0} V^{\otimes n}.$$

We have a coproduct $\Delta_{T(V)} : T(V) \longrightarrow T(V) \otimes T(V)$ defined by $\Delta_{T(V)}(1) := 1 \otimes 1$ and

$$\Delta_{T(V)}(v_1 \otimes \cdots \otimes v_n) := \sum_{k=0}^{n} (v_1 \otimes \cdots \otimes v_k) \otimes (v_{k+1} \otimes \cdots \otimes v_n).$$

for every $v_1, \ldots, v_n \in V$. Consider

$$\overline{T}(V) := \bigoplus_{n \geq 1} V^{\otimes n}.$$

We also have a coproduct on $\overline{T}(V)$, defined, for every $v_1, \ldots, v_n \in V$, by $\Delta_{\overline{T}(V)}(v_1) := 0$ and,

$$\Delta_{\overline{T}(V)}(v_1 \otimes \cdots \otimes v_n) := \sum_{k=1}^{n-1} (v_1 \otimes \cdots \otimes v_k) \otimes (v_{k+1} \otimes \cdots \otimes v_n).$$

We embed $V \otimes \Gamma(V) \subset \overline{T}(V)$. Note that

$$\Delta_{\overline{T}(V)}(V \otimes \Gamma(V)) \subset V \otimes \Gamma(V) \otimes \overline{\Gamma}(V).$$

By applying the embedding $(V^{\otimes n})^{\Sigma_n} \subset V \otimes (V^{\otimes n-1})^{\Sigma_{n-1}}$ for each $n \geq 2$, we have the inclusion $\overline{\Gamma}(V) \subset V \otimes \Gamma(V)$. We thus have obtained a coproduct

$$\Delta_{\Gamma\mathrm{Perm}} : \Gamma\mathrm{Perm}^c(V) \longrightarrow \Gamma\mathrm{Perm}^c(V) \otimes \Gamma\mathrm{Perm}^c(V).$$

We can also identify $\Delta_{\Gamma\mathrm{Perm}}$ with the composite

$$\begin{tikzcd} \Delta_{\Gamma\mathrm{Perm}} : V \otimes \Gamma(V) \arrow[r, "id \otimes \Delta_{T(V)}"] & V \otimes \Gamma(V) \otimes \Gamma(V) \arrow[d, two heads] & \\ & V \otimes \Gamma(V) \otimes \overline{\Gamma}(V) \arrow[r, hook] & (V \otimes \Gamma(V)) \otimes (V \otimes \Gamma(V)) \end{tikzcd}.$$

Lemma 2.2.11. *The morphism $\Delta_{\Gamma\mathrm{Perm}} : \Gamma\mathrm{Perm}^c(V) \longrightarrow \Gamma\mathrm{Perm}^c(V) \otimes \Gamma\mathrm{Perm}^c(V)$ satisfies the identities:*

$$(id \otimes \Delta_{\Gamma\mathrm{Perm}})\Delta_{\Gamma\mathrm{Perm}} = (\Delta \otimes id)\Delta_{\Gamma\mathrm{Perm}};$$

$$(id \otimes \Delta_{\Gamma\mathrm{Perm}})\Delta_{\Gamma\mathrm{Perm}} = (id \otimes \tau)(id \otimes \Delta_{\Gamma\mathrm{Perm}})\Delta_{\Gamma\mathrm{Perm}}.$$

Moreover, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Perm}^c(V) \arrow[r, "\Delta_{\mathrm{Perm}}"] \arrow[d, "id\otimes Tr"'] & \mathrm{Perm}^c(V)\otimes \mathrm{Perm}^c(V) \arrow[d, "(id\otimes Tr)\otimes(id\otimes Tr)"] \\
\Gamma\mathrm{Perm}^c(V) \arrow[r, "\Delta_{\Gamma\mathrm{Perm}}"'] & \Gamma\mathrm{Perm}^c(V)\otimes \Gamma\mathrm{Perm}^c(V)
\end{tikzcd}$$

Proof. The proof of this lemma comes from straightforward computations. □

Remark 2.2.12. *The relation* $(id\otimes\Delta_{\Gamma\mathrm{Perm}})\Delta_{\Gamma\mathrm{Perm}} = (id\otimes\tau)(id\otimes\Delta_{\Gamma\mathrm{Perm}})\Delta_{\Gamma\mathrm{Perm}}$ *implies that, for every* $k\geq 1$*,*

$$(\Delta_{\Gamma\mathrm{Perm}})^k(\Gamma\mathrm{Perm}^c(V)) \subset \Gamma\mathrm{Perm}^c(V)\otimes(\Gamma\mathrm{Perm}^c(V)^{\otimes k})^{\Sigma_k}.$$

As a consequence, since $(\Delta_{\overline{T}(V)})^k$ *reduces to the identity on* $V^{\otimes k+1}$*, and by definition of* $\Delta_{\Gamma\mathrm{Perm}}$*, we have the following commutative diagram:*

$$\begin{tikzcd}
\overline{T}(V) \arrow[r, "(\Delta_{\overline{T}(V)})^k"] \arrow[rr, bend left, "\pi_{V^{\otimes k+1}}"] & \overline{T}(V)^{\otimes k+1} \arrow[r, "\pi_V^{\otimes k+1}"] & V\otimes V^{\otimes k} \\
\Gamma\mathrm{Perm}^c(V) \arrow[u, hook] \arrow[r, "(\Delta_{\Gamma\mathrm{Perm}})^k"] \arrow[rr, bend right, "\pi_{V^{\otimes k+1}}"'] & \Gamma\mathrm{Perm}^c(V)\otimes(\Gamma\mathrm{Perm}^c(V)^{\otimes k})^{\Sigma_k} \arrow[u, hook] \arrow[r, "\pi_V^{\otimes k+1}"] & V\otimes(V^{\otimes k})^{\Sigma_k} \arrow[u, hook]
\end{tikzcd},$$

where, for every $k\geq 0$*, we denote by* $\pi_{V^{\otimes k}}:\overline{T}(V)\longrightarrow V^{\otimes k}$ *the projection onto* $V^{\otimes k}$*.*

Definition 2.2.13. *An endomorphism* d *of* $\Gamma\mathrm{Perm}^c(V)$ *is called a* coderivation *if it satisfies*

$$\Delta_{\Gamma\mathrm{Perm}}d = (d\otimes id + id\otimes d)\Delta_{\Gamma\mathrm{Perm}}.$$

We let $Coder(\Gamma\mathrm{Perm}^c(V))$ *to be the* $\mathbb{K}$*-module spanned by coderivations.*

Our goal is to prove that any coderivation is characterized by its composite with π_V. We rely on the following definition.

Definition 2.2.14. *Let* $w, v_1,\ldots,v_r\in V$*. We define* $\mathrm{Sh}: T(V)\otimes V\longrightarrow T(V)$ *by*

$$\mathrm{Sh}(v_1\otimes\cdots\otimes v_n;w) = \sum_{i=0}^{n} \pm v_1\otimes\cdots\otimes v_i\otimes w\otimes v_{i+1}\otimes\cdots\otimes v_n$$

where the sign is given by the permutation $v_1\otimes\cdots\otimes v_n\otimes w\mapsto \pm v_1\otimes\cdots\otimes v_i\otimes w\otimes v_{i+1}\otimes\cdots\otimes v_n$ *for every* $0\leq i\leq n$*. We also define analogously* $\mathrm{Sh}: V\otimes T(V)\longrightarrow T(V)$*.*

We immediately see that $\mathrm{Sh}(\Gamma(V)\otimes V)\subset\Gamma(V)$.

Proposition 2.2.15. *The map*

$$\begin{array}{ccc} Coder(\Gamma\mathrm{Perm}^c(V)) & \longrightarrow & \mathrm{Hom}(\Gamma\mathrm{Perm}^c(V), V) \\ d & \longmapsto & \pi_V\circ d \end{array}$$

is a bijection. We denote by $\widetilde{\Psi}$ *its inverse. If we consider the inverse* Ψ *given in the proof of Proposition 2.2.5, then* $\widetilde{\Psi}$ *is compatible with* Ψ *in the following sense. Let* $\widetilde{l} \in \mathrm{Hom}(\Gamma\mathrm{Perm}^c(V), V)$*. We define* $l \in \mathrm{Hom}(\mathrm{Perm}^c(V), V)$ *by the composite*

$$l : \mathrm{Perm}^c(V) \xrightarrow{id\otimes Tr} \Gamma\mathrm{Perm}^c(V) \xrightarrow{\widetilde{l}} V\ .$$

Then the following diagram is commutative:

$$\begin{tikzcd} \Gamma\mathrm{Perm}^c(V) \arrow[r, "\widetilde{\Psi}(\widetilde{l})"] & \Gamma\mathrm{Perm}^c(V) \\ \mathrm{Perm}^c(V) \arrow[u, "id\otimes Tr"] \arrow[r, "\Psi(l)"'] & \mathrm{Perm}^c(V) \arrow[u, "id\otimes Tr"'] \end{tikzcd}\ .$$

Proof. Let $\widetilde{l} : \Gamma\mathrm{Perm}^c(V) \longrightarrow V$ be a morphism. We define an endomorphism $\widetilde{\Psi}(\widetilde{l})$ of $\Gamma\mathrm{Perm}^c(V)$ by the sum of the composite

$$\widetilde{\Psi}_1(\widetilde{l}) : V \otimes \Gamma(V) \xrightarrow{id\otimes\Delta_{T(V)}} V \otimes \Gamma(V) \otimes \Gamma(V) \xrightarrow{\widetilde{l}\otimes id} V \otimes \Gamma(V)$$

and of the composite

$$\begin{tikzcd} \widetilde{\Psi}(\widetilde{l})_2 : V \otimes \Gamma(V) \arrow[r, "\Delta"] & V \otimes \Gamma(V) \otimes \Gamma(V) \arrow[d, "id\otimes\widetilde{l}"] & \\ & V \otimes \Gamma(V) \otimes V \arrow[r, "id\otimes\mathrm{Sh}(-;-)"] & V \otimes \Gamma(V) \end{tikzcd}\ .$$

Let $x \in V$ and $Y \in \Gamma(V)$. We write the coproduct $\Delta_{\Gamma\mathrm{Perm}}(x\otimes Y)$ by using the Sweedler notation without the sum symbol, as

$$\Delta_{\Gamma\mathrm{Perm}}(x \otimes Y) = (x \otimes Y_{(1)}) \otimes (x_{(2)} \otimes Y_{(2)}),$$

where $x_{(2)} \in V$ and $Y_{(1)}, Y_{(2)} \in \Gamma(V)$. Note that we have

$$(id \otimes \Delta_{T(V)})(x \otimes Y) = (x \otimes Y_{(1)}) \otimes (x_{(2)} \otimes Y_{(2)}) + (x \otimes Y) \otimes 1.$$

Then, by definition of $\widetilde{\Psi}_1(\widetilde{l})$ and $\widetilde{\Psi}_2(\widetilde{l})$, we have

$$\widetilde{\Psi}_1(\widetilde{l})(x \otimes Y) = \widetilde{l}(x \otimes Y_{(1)}) \otimes (x_{(2)} \otimes Y_{(2)}) + \widetilde{l}(x \otimes Y) \otimes 1$$

and

$$\widetilde{\Psi}_2(\widetilde{l})(x \otimes Y) = \pm x \otimes \mathrm{Sh}(Y_{(1)}; \widetilde{l}(x_{(2)} \otimes Y_{(2)})),$$

so that

$$\widetilde{\Psi}(\widetilde{l})(x \otimes Y) = \widetilde{l}(x \otimes Y_{(1)}) \otimes (x_{(2)} \otimes Y_{(2)}) + \widetilde{l}(x \otimes Y) \otimes 1 \pm x \otimes \mathrm{Sh}(Y_{(1)}; \widetilde{l}(x_{(2)} \otimes Y_{(2)})).$$

We set

$$\Delta_{\Gamma\mathrm{Perm}}(x \otimes Y_{(1)}) = (x \otimes Y_{(11)}) \otimes (x_{(1)} \otimes Y_{(12)});$$

$$\Delta_{\Gamma\mathrm{Perm}}(x_{(2)}\otimes Y_{(2)})=(x_{(2)}\otimes Y_{(21)})\otimes(x_{(22)}\otimes Y_{(22)}).$$

We then have

$$\begin{aligned}
(\widetilde{\Psi}(\widetilde{l})\otimes id)\Delta_{\Gamma\mathrm{Perm}}(x\otimes Y) &= \widetilde{l}(x\otimes Y_{(11)})\otimes(x_{(1)}\otimes Y_{(12)})\otimes(x_{(2)}\otimes Y_{(2)})\\
&\quad +\widetilde{l}(x\otimes Y_{(1)})\otimes 1\otimes(x_{(2)}\otimes Y_{(2)})\\
&\quad \pm(x\otimes \mathrm{Sh}(Y_{(11)};\widetilde{l}(x_{(1)}\otimes Y_{(12)})))\otimes(x_{(2)}\otimes Y_{(2)});\\
(id\otimes\widetilde{\Psi}(\widetilde{l}))\Delta_{\Gamma\mathrm{Perm}}(x\otimes Y) &= \pm(x\otimes Y_{(1)})\otimes(\widetilde{l}(x_{(2)}\otimes Y_{(21)})\otimes(x_{(22)}\otimes Y_{(22)}))\\
&\quad \pm(x\otimes Y_{(1)})\otimes(x_{(2)}\otimes \mathrm{Sh}(Y_{(21)};\widetilde{l}(x_{(22)}\otimes Y_{(22)})))\\
&\quad \pm(x\otimes Y_{(1)})\otimes\widetilde{l}(x_{(2)}\otimes Y_{(2)})\otimes 1.
\end{aligned}$$

We now compute $\Delta_{\Gamma\mathrm{Perm}}\widetilde{\Psi}(\widetilde{l})(x\otimes Y)$. The term $\Delta_{\Gamma\mathrm{Perm}}\widetilde{\Psi}_1(\widetilde{l})(x\otimes Y)$ gives

$$\widetilde{l}(x\otimes Y_{(1)})\otimes(x_{(2)}\otimes Y_{(21)})\otimes(x_{(22)}\otimes Y_{(22)})+\widetilde{l}(x\otimes Y_{(1)})\otimes 1\otimes(x_{(2)}\otimes Y_{(2)}).$$

By using the first identity of Lemma 2.2.11 which gives

$$(x\otimes Y_{(11)})\otimes(x_{(1)}\otimes Y_{(12)})\otimes(x_{(2)}\otimes Y_{(2)})=(x\otimes Y_{(1)})\otimes(x_{(2)}\otimes Y_{(21)})\otimes(x_{(22)}\otimes Y_{(22)}),$$

we have that $\Delta_{\Gamma\mathrm{Perm}}\widetilde{\Psi}_1(\widetilde{l})(x\otimes Y)$ is

$$\widetilde{l}(x\otimes Y_{(11)})\otimes(x_{(1)}\otimes Y_{(12)})\otimes(x_{(2)}\otimes Y_{(2)})+\widetilde{l}(x\otimes Y_{(1)})\otimes 1\otimes(x_{(2)}\otimes Y_{(2)}),$$

which is exactly the first two lines occurring in $(\widetilde{\Psi}(\widetilde{l})\otimes id)\Delta_{\Gamma\mathrm{Perm}}(x\otimes Y)$. The term $\Delta_{\Gamma\mathrm{Perm}}\widetilde{\Psi}_2(\widetilde{l})(x\otimes Y)$ gives

$$\begin{gathered}
\pm(x\otimes \mathrm{Sh}(Y_{(11)};\widetilde{l}(x_{(2)}\otimes Y_{(2)})))\otimes(x_{(1)}\otimes Y_{(12)})\\
\pm(x\otimes Y_{(11)})\otimes(\widetilde{l}(x_{(2)}\otimes Y_{(2)})\otimes Y_{(12)})\\
\pm(x\otimes Y_{(11)})\otimes(x_{(1)}\otimes \mathrm{Sh}(Y_{(12)};\widetilde{l}(x_{(2)}\otimes Y_{(2)})))\\
\pm(x\otimes Y_{(1)})\otimes(\widetilde{l}(x_{(2)}\otimes Y_{(2)})\otimes 1).
\end{gathered}$$

From the second formula given in Lemma 2.2.11, which gives

$$(x\otimes Y_{(11)})\otimes(x_{(1)}\otimes Y_{(12)})\otimes(x_{(2)}\otimes Y_{(2)})=\pm(x\otimes Y_{(11)})\otimes(x_{(2)}\otimes Y_{(2)})\otimes(x_{(1)}\otimes Y_{(12)}),$$

we obtain that $\Delta_{\Gamma\mathrm{Perm}}\widetilde{\Psi}_2(\widetilde{l})(x\otimes Y)$ is given by

$$\begin{gathered}
\pm(x\otimes \mathrm{Sh}(Y_{(11)};\widetilde{l}(x_{(1)}\otimes Y_{(12)})))\otimes(x_{(2)}\otimes Y_{(2)})\\
\pm(x\otimes Y_{(11)})\otimes(\widetilde{l}(x_{(1)}\otimes Y_{(12)})\otimes Y_{(2)})\\
\pm(x\otimes Y_{(11)})\otimes(x_{(2)}\otimes \mathrm{Sh}(Y_{(2)};\widetilde{l}(x_{(1)}\otimes Y_{(12)})))\\
\pm(x\otimes Y_{(1)})\otimes(\widetilde{l}(x_{(2)}\otimes Y_{(2)})\otimes 1).
\end{gathered}$$

The first line is the remaining term in $(\widetilde{\Psi}(\widetilde{l})\otimes id)\Delta_{\Gamma\mathrm{Perm}}(x\otimes Y)$, while the remaining lines give $(id\otimes\widetilde{\Psi}(\widetilde{l}))\Delta_{\Gamma\mathrm{Perm}}(x\otimes Y)$ when using again the first formula of Lemma 2.2.11. We thus have proved that $\widetilde{\Psi}(\widetilde{l})\in Coder(\Gamma\mathrm{Perm}^c(V))$, and $\pi_V\circ\widetilde{\Psi}(\widetilde{l})=\widetilde{l}$. We now prove that $\widetilde{\Psi}(\widetilde{l})$ is the only coderivation Q such that $\pi_V \circ Q = l$. We use Remark 2.2.12, which gives

$$\pi_{V^{\otimes k+1}} Q = \pi_V^{\otimes k+1} \Delta^k_{\Gamma\mathrm{Perm}} Q = \sum_{i=1}^{k+1} (\pi_V^{\otimes i-1} \otimes \widetilde{l} \otimes \pi_V^{\otimes k-i+1}) \Delta^k_{\Gamma\mathrm{Perm}},$$

which proves that Q is fully determined by $\widetilde{l}$. We thus have $Q = \widetilde{\Psi}(\widetilde{l})$, and then that Ψ is the desired bijection. We now prove the commutativity of the diagram. We note that, by definition of $\Psi(l), \widetilde{\Psi}(\widetilde{l})$ and l, the following diagram is commutative:

$$\begin{tikzcd}
\Gamma\mathrm{Perm}^c(V) \arrow[r, "id\otimes\Delta_{T(V)}"'] \arrow[rr, bend left, "\widetilde{\Psi}_1(\widetilde{l})"] & \Gamma\mathrm{Perm}^c(V)\otimes\Gamma(V) \arrow[r, "\widetilde{l}\otimes id"'] & \Gamma\mathrm{Perm}^c(V) \\
\mathrm{Perm}^c(V) \arrow[u, "id\otimes Tr"] \arrow[r, "id\otimes\Delta_{\mathcal{S}(V)}"] \arrow[rr, bend right, "\Psi_1(l)"'] & \mathrm{Perm}^c(V)\otimes\mathcal{S}(V) \arrow[u, "id\otimes Tr\otimes Tr"] \arrow[r, "l\otimes id"] & \mathrm{Perm}^c(V) \arrow[u, "id\otimes Tr"']
\end{tikzcd} .$$

By Lemma 2.2.11, and by the formula

$$Tr(v_1 \cdots v_n \cdot w) = \mathrm{Sh}(Tr(v_1 \cdots v_n); w),$$

we also obtain the following commutative diagram:

$$\begin{tikzcd}
\Gamma\mathrm{Perm}^c(V) \arrow[r, "\Delta_{\Gamma\mathrm{Perm}}"'] \arrow[rrr, bend left, "\widetilde{\Psi}_2(\widetilde{l})"] & \Gamma\mathrm{Perm}^c(V)\otimes\Gamma\mathrm{Perm}^c(V) \arrow[r, "id\otimes\widetilde{l}"'] & \Gamma\mathrm{Perm}^c(V)\otimes V \arrow[r, "id\otimes Sh(-;-)"'] & \Gamma\mathrm{Perm}^c(V) \\
\mathrm{Perm}^c(V) \arrow[u, "id\otimes Tr"] \arrow[r, "\Delta_{\mathrm{Perm}}"] \arrow[rrr, bend right, "\Psi_2(l)"'] & \mathrm{Perm}^c(V)\otimes\mathrm{Perm}^c(V) \arrow[u, "id\otimes Tr\otimes id\otimes Tr"] \arrow[r, "id\otimes l"] & \mathrm{Perm}^c(V)\otimes V \arrow[u, "id\otimes Tr\otimes id"'] \arrow[r, two heads] & \mathrm{Perm}^c(V) \arrow[u, "id\otimes Tr"']
\end{tikzcd}$$

which proves the theorem. □

Definition 2.2.16. *We define the category $\Gamma\Lambda\mathcal{PL}_\infty$ with as objects the pairs (V,Q) where V is a graded $\mathbb{K}$-module and Q a coderivation of degree -1 on $\Gamma\mathrm{Perm}^c(V)$ such that $Q^2 = 0$; a morphism $\phi : (V,Q) \longrightarrow (V',Q')$ is a morphism of coalgebras $\phi : \Gamma\mathrm{Perm}^c(V) \longrightarrow \Gamma\mathrm{Perm}^c(V')$ which commutes with the coderivations Q and Q'.*

We usually denote a morphism $\phi : (V,Q) \longrightarrow (V',Q')$ by $\phi : V \rightsquigarrow V'$ when there is no ambiguity on Q and Q', and call it an *∞-morphism.*

If $\phi : \Gamma\mathrm{Perm}^c(V) \longrightarrow \Gamma\mathrm{Perm}^c(V')$ is a morphism of graded $\mathbb{K}$-modules, then we set, for all $k,n \geq 0$,

$$\phi_k : V \otimes (V^{\otimes k})^{\Sigma_k} \hookrightarrow \Gamma\mathrm{Perm}^c(V) \xrightarrow{\phi} \Gamma\mathrm{Perm}^c(V');$$

$$\phi^n : \Gamma\mathrm{Perm}^c(V) \xrightarrow{\phi} \Gamma\mathrm{Perm}^c(V') \xrightarrow{\pi_{V^{\otimes n+1}}} V' \otimes (V'^{\otimes n})^{\Sigma_n}.$$

Using these notations, a degree -1 coderivation Q on $\Gamma\mathrm{Perm}^c(V)$ is such that $Q^2 = 0$ if and only if for all $n \geq 0$,

$$\sum_{k=0}^{n} Q_k^0 Q_n^k = 0.$$

In particular, Q_0^0 is a differential on V. From now on, we endow $V \in \Gamma\Lambda\mathcal{PL}_\infty$ with the structure of a dg $\mathbb{K}$-module with differential $d = Q_0^0$. We also have that a morphism of graded $\mathbb{K}$-modules $\phi : \Gamma\mathrm{Perm}^c(V) \longrightarrow \Gamma\mathrm{Perm}^c(V')$ is a morphism of coalgebras if and only if

$$\Delta_{\Gamma\mathrm{Perm}}\phi^n = \sum_{p+q=n-1} (\phi^p \otimes \phi^q)\Delta_{\Gamma\mathrm{Perm}}.$$

Proposition 2.2.17. *Every ∞-morphism $\phi : V \longrightarrow W$ in $\Gamma\Lambda\mathcal{PL}_\infty$ is fully determined by the composite $\phi^0 = \pi_W \circ \phi$.*

Proof. Let ϕ be an ∞-morphism. We have that

$$\phi^k = \pi_W^{\otimes k+1}\Delta_{\Gamma\mathrm{Perm}}^k\phi^k = (\phi^0)^{\otimes k}\Delta_{\Gamma\mathrm{Perm}}^k,$$

which gives, for every $v \in \Gamma\mathrm{Perm}^c(V)$,

$$\phi^k(v) = \phi^0(v_{(1)}) \otimes \cdots \otimes \phi^0(v_{(k)})$$

where we use the Sweedler notation in the coalgebra $\Gamma\mathrm{Perm}^c(V)$. We then see that ϕ is fully determined by ϕ^0. $\square$

Remark 2.2.18. *This proposition implies that giving an ∞-morphism $\phi : V \rightsquigarrow W$ is equivalent to giving a morphism $\phi^0 : \Gamma\mathrm{Perm}^c(V) \longrightarrow W$ such that the morphism $\phi : \Gamma\mathrm{Perm}^c(V) \longrightarrow \Gamma\mathrm{Perm}^c(W)$ constructed in Proposition 2.2.17 satisfies*

$$\sum_{k=0}^{n} (Q')_k^0 \phi_n^k = \sum_{k=0}^{n} \phi_k^0 Q_n^k$$

for every $n \geq 0$. In particular, $\phi_0^0 : V \longrightarrow W$ is a morphism of dg $\mathbb{K}$-modules.

Definition 2.2.19. *An ∞-morphism $\phi : V \rightsquigarrow W$ is* strict *if $\phi_k^0 = 0$ for all $k \geq 1$.*

Equivalently, a strict morphism $\phi : V \longrightarrow W$ is the data of a morphism of dg $\mathbb{K}$-modules $\phi : V \longrightarrow W$ such that

$$(Q')_n^0 \phi^{\otimes n+1} = \phi Q_n^0$$

for every $n \geq 0$.

2.2.3 Symmetric weighted braces and Maurer-Cartan elements in $\widehat{\Gamma\Lambda\mathcal{PL}_\infty}$

In this subsection, we define weighted brace operations for $\Gamma\Lambda\mathcal{PL}_\infty$-algebras, and prove that giving a structure of a $\Gamma\Lambda\mathcal{PL}_\infty$-algebra is equivalent to giving such operations. These operations will be analogue to the operations given in [Ver23, Theorem

2.6] for $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebras. We also define the notion of Maurer-Cartan element in complete $\Gamma\Lambda\mathcal{PL}_\infty$-algebras.

We first need an explicit basis of $\Gamma\mathrm{Perm}^c(V)$. We use the same arguments as in [Ver23, §2.1.1]. Let $\mathcal{B}$ be a basis of V composed of homogeneous elements. For every $n \geq 0$, this gives a basis on $V^{\otimes n}$ which we denote by $\mathcal{B}^{\otimes n}$. We consider the action of Σ_n on $\mathcal{B}^{\otimes n}$ by permutation of the factors without the Koszul sign rule. For every $\mathfrak{t} \in \mathcal{B}^{\otimes n}$, we denote by $X_\mathfrak{t}$ the orbit of $\mathfrak{t}$ under this action. We then have the unequivariant identity
$$V^{\otimes n} = \bigoplus_{\mathfrak{t}\in\mathcal{B}^{\otimes n}/\Sigma_n} \mathbb{K}[X_\mathfrak{t}].$$
For every $\mathfrak{t} \in \mathcal{B}^{\otimes n}$, we set $\mathbb{K}[X_\mathfrak{t}]^\pm = \mathbb{K}[X_\mathfrak{t}]$ with underlying action
$$\sigma \cdot x = \varepsilon(\sigma, x)x$$
for every $\sigma \in \Sigma_n$ and $x \in X_\mathfrak{t}$, where we denote by $\varepsilon(\sigma, x) \in \mathbb{K}$ the Koszul sign which appears after the action of σ on x. We then have the identification of Σ_n-representations:
$$V^{\otimes n} = \bigoplus_{\mathfrak{t}\in\mathcal{B}^{\otimes n}/\Sigma_n} \mathbb{K}[X_\mathfrak{t}]^\pm.$$
Let $(\mathcal{B}^{\otimes n})^s$ be the subset of $\mathcal{B}^{\otimes n}$ given by elements $\mathfrak{t} \in \mathcal{B}^{\otimes n}$ such that there exists $\sigma \in \mathrm{Stab}_{\Sigma_n}(\mathfrak{t})$ with $\varepsilon(\sigma, \mathfrak{t}) \neq 1$. We set $(\mathcal{B}^{\otimes n})^r = \mathcal{B}^{\otimes n} \setminus (\mathcal{B}^{\otimes n})^s$. Note that, if $char(\mathbb{K}) = 2$, then $(\mathcal{B}^{\otimes n})^r = \mathcal{B}^{\otimes n}$, else, the subset $(\mathcal{B}^{\otimes n})^r$ is given by tensors of the form $x_1^{\otimes r_1} \otimes \cdots \otimes x_n^{\otimes r_n}$ with $x_1, \ldots, x_n \in \mathcal{B}$ pairwise distinct and $r_1, \ldots, r_n \geq 0$ such that if x_i has an odd degree for some i, then $r_i = 1$. We let $\mathcal{S}^r(V)$ to be given by the projections of $(\mathcal{B}^{\otimes n})^r$ on $\mathcal{S}(V)$.

Proposition 2.2.20. *The map $\mathcal{O} : \mathcal{S}^r(V) \longrightarrow \Gamma(V)$ defined by*
$$\mathcal{O}(x_1 \cdots x_n) = \sum_{\sigma\in\Sigma_n/\mathrm{Stab}_{\Sigma_n}(x_1\otimes\cdots\otimes x_n)} \pm x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}$$
is an isomorphism.

Proof. It is the same arguments as in [Ver23, Proposition 2.5]. □

In the following, in order to handle both the cases $char(\mathbb{K}) = 2$ and $char(\mathbb{K}) \neq 2$, when taking elements with associated weights, we will tacitly suppose that if $char(\mathbb{K}) \neq 2$, then all odd degree elements will have an associated weight equal to 1.

Lemma 2.2.21. *Let $x \in V, y_1, \ldots, y_n \in \mathcal{B}$ and $r_1, \ldots, r_n \geq 0$. Then*
$$\Delta_{\Gamma\mathrm{Perm}}(x\otimes\mathcal{O}(y_1^{\otimes r_1} \cdots y_n^{\otimes r_n})) = \sum_{k=1}^{n} \sum_{\substack{p_i+q_i=r_i, i\neq k\\ p_k+q_k=r_k-1}} \pm(x\otimes\mathcal{O}(y_1^{\otimes p_1} \cdots y_n^{\otimes p_n}))\otimes(y_k\otimes\mathcal{O}(y_1^{\otimes q_1} \cdots y_n^{\otimes q_n})),$$
where the sign is yielded by the shuffle
$$x \otimes y_1^{\otimes r_1} \otimes \cdots \otimes y_n^{\otimes r_n} \longmapsto \pm x \otimes y_1^{\otimes p_1} \otimes \cdots \otimes y_n^{\otimes p_n} \otimes y_k \otimes y_1^{\otimes q_1} \otimes \cdots \otimes y_n^{\otimes q_n}.$$

Proof. Straightforward computations. □

Theorem 2.2.22. *Let $V \in \Gamma\Lambda\mathcal{PL}_\infty$. Then V comes equipped with operations, called* weighted braces, *which have the following form.*

- *If $char(\mathbb{K}) = 2$, then weighted braces are maps*

$$-\{\!\!\{-,\dots,-\}\!\!\}_{r_1,\dots,r_n} : V^{\times n+1} \longrightarrow V,$$

defined for any collections of integers $r_1,\dots,r_n \geq 0$, which preserve the grading in the sense that

$$V_k\{\!\!\{V_{k_1},\dots,V_{k_n}\}\!\!\}_{r_1,\dots,r_n} \subset V_{k+k_1r_1+\cdots+k_nr_n}.$$

- *If $char(\mathbb{K}) \neq 2$, by setting $V^{ev} = \bigoplus_{n\in\mathbb{Z}} V_{2n}$ and $V^{odd} = \bigoplus_{n\in\mathbb{Z}} V_{2n+1}$, weighted brace are maps*

$$-\{\!\!\{\underbrace{-,\dots,-}_{p},\underbrace{-,\dots,-}_{q}\}\!\!\}_{r_1,\dots,r_p,1,\dots,1} : V \times (V^{ev})^{\times p} \times (V^{odd})^{\times q} \longrightarrow V,$$

defined for any collection of integers $p,q,r_1,\dots,r_n \geq 0$ which preserve the grading.

In addition, in both cases, the weighted brace operations satisfy the following formulas:

(i) $x\{\!\!\{y_{\sigma(1)},\dots,y_{\sigma(n)}\}\!\!\}_{r_{\sigma(1)},\dots,r_{\sigma(n)}} = \pm x\{\!\!\{y_1,\dots,y_n\}\!\!\}_{r_1,\dots,r_n}$,

(ii) $$x\{\!\!\{y_1,\dots,y_{i-1},y_i,y_{i+1},\dots,y_n\}\!\!\}_{r_1,\dots,r_{i-1},0,r_{i+1},\dots,r_n}$$
$$= x\{\!\!\{y_1,\dots,y_{i-1},y_{i+1},\dots,y_n\}\!\!\}_{r_1,\dots,r_{i-1},r_{i+1},\dots,r_n},$$

(iii) $x\{\!\!\{y_1,\dots,\lambda y_i,\dots,y_n\}\!\!\}_{r_1,\dots,r_i,\dots,r_n} = \lambda^{r_i} x\{\!\!\{y_1,\dots,y_i,\dots,y_n\}\!\!\}_{r_1,\dots,r_i,\dots,r_n}$,

(iv) $$x\{\!\!\{y_1,\dots,y_i,y_i,\dots,y_n\}\!\!\}_{r_1,\dots,r_i,r_{i+1},\dots,r_n}$$
$$= \binom{r_i+r_{i+1}}{r_i} x\{\!\!\{y_1,\dots,y_i,\dots,y_n\}\!\!\}_{r_1,\dots,r_{i-1},r_i+r_{i+1},r_{i+2},\dots,r_n},$$

(v) $x\{\!\!\{y_1,\dots,y_i+\widetilde{y}_i,\dots,y_n\}\!\!\}_{r_1,\dots,r_i,\dots,r_n} = \sum_{s=0}^{r_i} x\{\!\!\{y_1,\dots,y_i,\widetilde{y}_i,\dots,y_n\}\!\!\}_{r_1,\dots,s,r_i-s,\dots,r_n}$,

(vi) $$\sum_{p_i+q_i=r_i} \pm x\{\!\!\{y_1,\dots,y_n\}\!\!\}_{p_1,\dots,p_n}\{\!\!\{y_1,\dots,y_n\}\!\!\}_{q_1,\dots,q_n}$$
$$+\sum_{k=1}^{n}\sum_{\substack{p_i+q_i=r_i, i\neq k\\ p_k+q_k=r_k-1}} \pm x\{\!\!\{y_k\{\!\!\{y_1,\dots,y_n\}\!\!\}_{p_1,\dots,p_n},y_1,\dots,y_n\}\!\!\}_{1,q_1,\dots,q_n} = 0.$$

In the converse direction, if a graded $\mathbb{K}$-module V admits such operations, then $V \in \Gamma\Lambda\mathcal{PL}_\infty$.

In particular, the operation $d(x) := x\{\!\!\{\}\!\!\}$ is a differential. We usually endow $V \in \Gamma\Lambda\mathcal{PL}_\infty$ with the structure of a dg $\mathbb{K}$-module with differential d.

Proof. Let $V \in \Gamma\Lambda\mathcal{PL}_\infty$. The strategy is the same as in [Ver23, Theorem 2.6] or [Ces18, Proposition 5.10]. Let $x, y_1, \ldots, y_n \in V$ be homogeneous elements, and $e_1, \ldots, e_n$ be formal elements with the same degrees as $y_1, \ldots, y_n$. We let E to be graded $\mathbb{K}$-module spanned by $Y_1, \ldots, Y_n$. Let $\psi : \Gamma(E) \longrightarrow \Gamma(V)$ be the morphism which sends the Y_i's to the y_i's. We immediately see that ψ is a morphism of coalgebras. We set

$$x\{\!\{y_1, \ldots, y_n\}\!\}_{r_1,\ldots,r_n} := Q^0_{\sum_i r_i}(x \otimes \psi\mathcal{O}(Y_1^{\otimes r_1} \cdots Y_n^{\otimes r_n})).$$

Formulas $(i)-(v)$ are consequences of straightforward computations. We prove formula (vi). Since ψ is a morphism of coalgebras, Lemma 2.2.21 gives

$$\begin{aligned} Q(x \otimes \psi\mathcal{O}(Y_1^{r_1} \cdots Y_n^{r_n})) = & \sum_{p_i+q_i=r_i} \pm Q^0(x \otimes \psi\mathcal{O}(Y_1^{p_1} \cdots Y_n^{p_n})) \otimes \psi(\mathcal{O}(Y_1^{q_1} \cdots Y_n^{q_n}))) \\ & + \sum_{k=1}^{n} \sum_{\substack{p_i+q_i=r_i, i\neq k \\ p_k+q_k=r_k-1}} \pm x \otimes \mathrm{Sh}(Q^0(y_k \otimes \psi\mathcal{O}(Y_1^{p_1} \cdots Y_n^{p_n})); \psi\mathcal{O}(Y_1^{q_1} \cdots Y_n^{q_n})). \end{aligned}$$

For fixed p_i's and q_i's, we have

$$Q^0(Q^0(x \otimes \psi\mathcal{O}(Y_1^{p_1} \cdots Y_n^{p_n})) \otimes \psi\mathcal{O}(Y_1^{q_1} \cdots Y_n^{q_n})) = x\{\!\{y_1, \ldots, y_n\}\!\}_{p_1,\ldots,p_n}\{\!\{y_1, \ldots, y_n\}\!\}_{q_1,\ldots,q_n},$$

by definition of weighted brace operations. Concerning the second line, for fixed p_i's, q_i's and k, let Z be a formal element with the same degree as $Q^0(y_k \otimes \psi\mathcal{O}(Y_1^{p_1}, \ldots, Y_n^{p_n}))$. We extend ψ to $\psi : \Gamma(E \oplus \mathbb{K}Z) \longrightarrow \Gamma(V)$ by sending Z to $Q^0(y_k \otimes \psi\mathcal{O}(Y_1^{p_1}, \ldots, Y_n^{p_n}))$. We then have

$$\mathrm{Sh}(Q^0(y_k \otimes \psi\mathcal{O}(Y_1^{p_1} \cdots Y_n^{p_n})); \psi\mathcal{O}(Y_1^{q_1} \cdots Y_n^{q_n})) = \psi\mathcal{O}(Z \cdot Y_1^{q_1} \cdots Y_n^{q_n}).$$

Taking the image under Q^0 thus gives

$$\begin{aligned} & Q^0(x \otimes \mathrm{Sh}(Q^0(y_k \otimes \psi\mathcal{O}(Y_1^{p_1} \cdots Y_n^{p_n})); \psi\mathcal{O}(Y_1^{q_1} \cdots Y_n^{q_n}))) \\ & \qquad = x\{\!\{y_k\{\!\{y_1, \ldots, y_n\}\!\}_{p_1,\ldots,p_n}, y_1, \ldots, y_n\}\!\}_{1,q_1,\ldots,q_n}. \end{aligned}$$

Since $Q^0Q = 0$, formula (vi) follows. We now prove the converse direction. Suppose that V is a dg $\mathbb{K}$-module equipped with operations $-\{\!\{-, \ldots, -\}\!\}_{r_1,\ldots,r_n}$ for all $r_1, \ldots, r_n \geq 0$ which satisfy the formulas given in the theorem. We pick a basis $\mathcal{B}$ of V composed of homogeneous elements. Let $x, y_1, \ldots, y_n \in \mathcal{B}$. For all $r_1, \ldots, r_n \geq 0$, we set

$$Q^0(x \otimes \mathcal{O}(y_1^{r_1} \cdots y_n^{r_n})) = x\{\!\{y_1, \ldots, y_n\}\!\}_{r_1,\ldots,r_n}$$

where we consider the orbit map $\mathcal{O}$ associated to the basis $\mathcal{B}$. By formulas $(iii) - (v)$ and the same computations as in [Ces18, Lemma 5.15], this definition does not depend on the choice of $\mathcal{B}$. Let $Q = \widetilde{\Psi}(Q^0)$ be the coderivation associated to $Q^0 \in \mathrm{Hom}(\Gamma\mathrm{Perm}^c(V), V)$ given by Proposition 2.2.15. We need to prove that $Q^2 = 0$, which is equivalent to prove that $Q^0Q = 0$. By Lemma 2.2.21, we have

$$Q(x \otimes \mathcal{O}(y_1^{r_1} \cdots y_n^{r_n})) = \sum_{p_i+q_i=r_i} \pm Q^0(x \otimes \mathcal{O}(y_1^{p_1} \cdots y_n^{p_n})) \otimes \mathcal{O}(y_1^{q_1} \cdots y_n^{q_n})$$
$$+ \sum_{k=1}^{n} \sum_{\substack{p_i+q_i=r_i, i\neq k \\ p_k+q_k=r_k-1}} \pm x \otimes \mathrm{Sh}(Q^0(y_k \otimes \mathcal{O}(y_1^{p_1} \cdots y_n^{p_n})); \mathcal{O}(y_1^{q_1} \cdots y_n^{q_n})).$$

Applying Q^0 to this identity gives $Q^0Q = 0$. □

Remark 2.2.23. *A strict morphism $\phi : V \longrightarrow W$ preserves the braces in the sense that*

$$\phi(x\{\!\{y_1, \ldots, y_n\}\!\}_{r_1,\ldots,r_n}) = \pm\phi(x)\{\!\{\phi(y_1), \ldots, \phi(y_n)\}\!\}_{r_1,\ldots,r_n},$$

where $\pm$ is produced by the commutation of ϕ with x and the $y_i^{\otimes r_i}$'s.

We aim to define the notion of a Maurer-Cartan element. To achieve this, we define the notion of a complete $\Gamma\Lambda\mathcal{PL}_\infty$-algebra.

Definition 2.2.24. *A* filtered $\Gamma\Lambda\mathcal{PL}_\infty$-algebra *is a $\Gamma\Lambda\mathcal{PL}_\infty$-algebra V endowed with a filtration $(F_nV)_{n\geq 1}$ such that*

$$F_mV\{\!\{F_{p_1}V, \ldots, F_{p_n}V\}\!\}_{r_1,\ldots,r_n} \subset F_{m+p_1r_1+\cdots+p_nr_n}V,$$

for all $m, p_1, \ldots, p_n \geq 1$ and $r_1, \ldots, r_n \geq 0$. An ∞-morphism $\phi : V \rightsquigarrow V'$ between two filtered $\Gamma\Lambda\mathcal{PL}_\infty$-algebras is an ∞-morphism such that

$$\phi_n^0(F_k(V^{\otimes n+1}) \cap \Gamma\mathrm{Perm}^c(V)) \subset F_k(V').$$

for every $k \geq 1$, where we consider the filtration associated to a tensor product (see the end of §2.1.1). A filtered $\Gamma\Lambda\mathcal{PL}_\infty$-algebra is complete *if the map $V \longrightarrow \lim_{n\geq 1} V/F_nV$ is an isomorphism.*

We denote by $\widehat{\Gamma\Lambda\mathcal{PL}_\infty}$ the category formed by complete filtered $\Gamma\Lambda\mathcal{PL}_\infty$-algebras with as morphisms the ∞-morphisms which preserve the filtrations.

Remark 2.2.25. *If V is a filtered $\Gamma\Lambda\mathcal{PL}_\infty$-algebra, then its completion $\widehat{V}$ admits the structure of a complete filtered $\Gamma\Lambda\mathcal{PL}_\infty$-algebra.*

Definition 2.2.26. *Let $V \in \widehat{\Gamma\Lambda\mathcal{PL}_\infty}$. A* Maurer-Cartan element *is an element $x \in V_0$ such that*

$$d(x) + \sum_{n\geq 1} x\{\!\{x\}\!\}_n = 0.$$

We denote by $\mathcal{MC}(V)$ the set composed of Maurer-Cartan elements.

Proposition 2.2.27. *Let $V, V' \in \widehat{\Gamma\Lambda\mathcal{PL}_\infty}$ and $\phi : V \rightsquigarrow V'$. Then ϕ induces a map*

$$\begin{array}{rccc} \mathcal{MC}(\phi) : & \mathcal{MC}(V) & \longrightarrow & \mathcal{MC}(V') \\ & x & \longmapsto & \sum_{n\geq 0} \phi_n^0(x \otimes x^{\otimes n}) \end{array}$$

such that $\mathcal{MC}(-) : \Gamma\widehat{\Lambda\mathcal{PL}_\infty} \longrightarrow \mathrm{Set}$ *is a functor. Moreover, if* ϕ_0^0 *is an isomorphism, then* $\mathcal{MC}(\phi)$ *is a bijection.*

Proof. Let $x \in \mathcal{MC}(V)$. We first prove that $y = \sum_{n\geq 0} \phi_n^0(x^{\otimes n+1}) \in \mathcal{MC}(V')$. We have

$$\sum_{m\geq 0}(Q')^0_m(y^{\otimes m+1}) = \sum_{m\geq 0}\sum_{k\geq m}(Q')^0_m\left(\sum_{p_0+\cdots+p_m=k-m}\phi^0_{p_0}(x^{\otimes p_0+1})\otimes\cdots\otimes\phi^0_{p_m}(x^{\otimes p_m+1})\right).$$

By using the proof of Proposition 2.2.17, we have

$$\begin{aligned}\sum_{m\geq 0}(Q')^0_m(y^{\otimes m+1}) &= \sum_{k\geq 0}\sum_{m=0}^{k}(Q')^0_m\phi_k^m(x^{\otimes k+1})\\ &= \sum_{k\geq 0}\sum_{m=0}^{k}\phi^0_m Q_k^m(x^{\otimes k+1})\\ &= \phi^0\left(\sum_{k\geq 0}Q_k(x^{\otimes k+1})\right).\end{aligned}$$

By using that $Q = \widetilde{\Psi}(Q^0)$ (see the proof of Proposition 2.2.15), we obtain

$$\sum_{k\geq 0}Q_k(x^{\otimes k+1}) = \sum_{q\geq 0}\left(\sum_{p\geq 0}Q_p^0(x^{\otimes p+1})\right)\otimes x^{\otimes q} \pm \sum_{p\geq 0} x\otimes \mathrm{Sh}\left(x^{\otimes p};\sum_{q\geq 0}Q_q^0(x^{\otimes q+1})\right) = 0$$

since $x \in \mathcal{MC}(V)$. The map $\mathcal{MC}(\phi)$ is thus well defined. Suppose now that ϕ_0^0 is an isomorphism, and let $y \in \mathcal{MC}(V')$. We search $x \in \mathcal{MC}(V)$ such that

$$\sum_{n\geq 0}\phi_n^0(x^{\otimes n+1}) = y,$$

which is equivalent to

$$x = (\phi_0^0)^{-1}\left(y - \sum_{n\geq 1}\phi_n^0(x^{\otimes n+1})\right).$$

We set $x_0 = (\phi_0^0)^{-1}(y)$. We define a Cauchy sequence $(x_k)_k$ by induction by

$$x_{k+1} = (\phi_0^0)^{-1}\left(y - \sum_{n\geq 1}\phi_n^0(x_k^{\otimes n+1})\right).$$

We denote by x its limit. We show that $x \in \mathcal{MC}(V)$. For every $W \in \Gamma\Lambda\mathcal{PL}_\infty$, we set

$$\mathcal{R}(w) = d(w) + \sum_{n\geq 1} w\{\!\{w\}\!\}_n$$

for every $w \in W_0$. We apply $\mathcal{R}$ on the identity $\sum_{n\geq 0}\phi_n^0(x^{\otimes n+1}) = y$, and use that

$y \in \mathcal{MC}(V')$:

$$\sum_{p\geq 0}(Q')^0_p\left(\left(\sum_{n\geq 0}\phi^0_n(x^{\otimes n+1})\right)^{\otimes p+1}\right) = 0.$$

This can be written as

$$\sum_{n\geq 0}\sum_{p=0}^{n}(Q')^0_p\phi^p_n(x^{\otimes n+1}) = 0.$$

Using that ϕ is a morphism in $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$, we obtain that

$$\sum_{n\geq 0}\sum_{p=0}^{n}\phi^0_pQ^p_n(x^{\otimes n+1}) = 0$$

which gives

$$\mathcal{R}(x) = -(\phi^0_0)^{-1}\left(\sum_{n\geq 1}\sum_{p=1}^{n}\phi^0_pQ^p_n(x^{\otimes n+1})\right).$$

We use the computation of Q as $Q = \widetilde{\Psi}(Q^0)$:

$$\mathcal{R}(x) = -(\phi^0_0)^{-1}\left(\sum_{n\geq 1}\sum_{p=1}^{n}\phi^0_p(Q^0_{n-p}(x^{\otimes n-p+1})\otimes x^{\otimes p} + x\otimes \mathrm{Sh}(x^{\otimes p-1};Q^0_{n-p}(x^{\otimes n-p+1})))\right).$$

We finally obtain

$$\mathcal{R}(x) = -(\phi^0_0)^{-1}\left(\sum_{p\geq 1}\phi^0_p(\mathcal{R}(x)\otimes x^{\otimes p} + x\otimes \mathrm{Sh}(x^{\otimes p-1};\mathcal{R}(x)))\right).$$

From this identity, and because ϕ preserves the filtrations on V and V', we have that if $\mathcal{R}(x) \in F_kV$ for some $k \geq 1$, then $\mathcal{R}(x) \in F_{k+1}V$. Since $\mathcal{R}(x) \in F_1V$, it follows that $\mathcal{R}(x) \in \bigcap_{k\geq 1}F_kV = 0$ so that $x \in \mathcal{MC}(V)$, and $\mathcal{MC}(\phi)(x) = y$ by construction. The map $\mathcal{MC}(\phi)$ is then surjective. We now prove that it is injective. Suppose that there exists $x_1, x_2 \in \mathcal{MC}(V)$ such that $\mathcal{MC}(\phi)(x_1) = \mathcal{MC}(\phi)(x_2)$. Then

$$x_1 - x_2 = (\phi^0_0)^{-1}\left(\sum_{n\geq 1}(x_2^{\otimes n+1} - x_1^{\otimes n+1})\right).$$

Suppose that $x_1 - x_2 \in F_kV$ for some $k \geq 1$. Then there exists $\alpha_k \in F_kV$ such that $x_1 = x_2 + \alpha_k$. By definition of the filtration on tensor products, and because $x_2 \in F_1V$, for every $n \geq 0$, we have $x_1^{\otimes n+1} = x_2^{\otimes n+1} + \alpha'_k$ where $\alpha'_k \in F_{k+1}V$ so that $x_2^{\otimes n+1} - x_1^{\otimes n+1} \in F_{k+1}V$. Since that ϕ preserves the filtrations, this implies $x_1 - x_2 \in F_{k+1}V$. We thus have $x_1 = x_2$, so that $\mathcal{MC}(\phi)$ is injective. □

2.2.4 Pre-Lie algebras up to homotopy with divided powers and $\Gamma\Lambda\mathcal{PL}_\infty$

In this subsection, we show that giving a structure of a $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebra is equivalent to giving the structure of a $\Gamma\Lambda\mathcal{PL}_\infty$-algebra up to a shift.

Let L be a dg $\mathbb{K}$-module. We make explicit a choice of a basis for $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, L)$ so that we can apply [Ver23, Lemma 2.3]. Let $\mathcal{B}$ be a basis of L. As a basis for $B^c(\Lambda^{-1}\mathrm{Perm}^\vee)(n)$, we consider tree monomials in $\mathcal{F}(\Sigma^{-1}\Lambda^{-1}\overline{\mathrm{Perm}^\vee})(n)$ with as vertices elements of the form $\Sigma^{j-2}e_i^j$ where $j \geq 2$ and $1 \leq i \leq j$ (see [DK10, §3.1] for a definition of these trees, or also Definition 2.6.4). We denote by $\mathcal{TM}(n)$ the set of tree monomials with n inputs. This gives a basis of $\mathcal{P}re\mathcal{L}ie_\infty(n) \otimes L^{\otimes n}$ which we denote by $\mathcal{TM}(n) \otimes \mathcal{B}^{\otimes n}$. We consider the action of Σ_n on $\mathcal{TM}(n)$ given by the action of Σ_n on $B^c(\Lambda^{-1}\mathrm{Perm}^\vee)$ where we omit the Koszul sign rule obtained after using the equivariance axioms for trees in $B^c(\Lambda^{-1}\mathrm{Perm}^\vee)$ in order to obtain a tree monomial. We also consider the action of Σ_n on $\mathcal{B}^{\otimes n}$ by permutations. We deduce an action of Σ_n on $\mathcal{TM}(n) \otimes \mathcal{B}^{\otimes n}$ defined as the diagonal action which uses the two previous actions of Σ_n on $\mathcal{TM}(n)$ and $\mathcal{B}^{\otimes n}$. Given such an action, we can write

$$\mathcal{P}re\mathcal{L}ie_\infty(n) \otimes L^{\otimes n} = \bigoplus_{\mathfrak{t} \in (\mathcal{TM}(n)\otimes\mathcal{B}^{\otimes n})/\Sigma_n} \mathbb{K}[X_\mathfrak{t}]$$

where we denote by $X_\mathfrak{t}$ the orbit of the element $\mathfrak{t} \in \mathcal{TM}(n) \otimes \mathcal{B}^{\otimes n}$ under the above action. Now, for every $\mathfrak{t} \in \mathcal{TM}(n)\otimes\mathcal{B}^{\otimes n}, \sigma \in \Sigma_n$ and $x \in X_\mathfrak{t}$, we denote by $\varepsilon(\sigma, x) \in \mathbb{K}$ the Koszul sign which appears after the action of σ on x, using the usual actions of Σ_n on $\mathcal{TM}(n)$ and $\mathcal{B}^{\otimes n}$. We define the Σ_n-representation $\mathbb{K}[X_\mathfrak{t}]^\pm$ as $\mathbb{K}[X_\mathfrak{t}]$ endowed with the Σ_n-action given by

$$\sigma \cdot x^\pm = \varepsilon(\sigma, x)(\sigma \cdot x)^\pm.$$

We obtain the following identification of Σ_n-representations:

$$\mathcal{P}re\mathcal{L}ie_\infty(n) \otimes L^{\otimes n} = \bigoplus_{\mathfrak{t} \in (\mathcal{TM}(n)\otimes\mathcal{B}^{\otimes n})/\Sigma_n} \mathbb{K}[X_\mathfrak{t}]^\pm.$$

Lemma 2.2.28. *For every $n \geq 0$, let $(\mathcal{TM}(n) \otimes \mathcal{B}^{\otimes n})^r$ be the subset of $\mathcal{TM}(n) \otimes \mathcal{B}^{\otimes n}$ formed by elements x such that, if $\sigma \cdot x = x$ for some $\sigma \in \Sigma_n$, then $\varepsilon(\sigma, x) = 1$. Let $\mathcal{S}^r(\mathcal{P}re\mathcal{L}ie_\infty, L)$ be the subspace of $\mathcal{S}(\mathcal{P}re\mathcal{L}ie_\infty, L)$ given by these elements. Then we have an isomorphism*

$$\mathcal{O} : \mathcal{S}^r(\mathcal{P}re\mathcal{L}ie_\infty, L) \longrightarrow \Gamma(\mathcal{P}re\mathcal{L}ie_\infty, L).$$

Proof. This comes from the previous analysis and [Ver23, Lemma 2.3]. See also the proof of [Ver23, Proposition 2.5]. □

Lemma 2.2.29. *Let L be a dg $\mathbb{K}$-module. Denote by $\mu : \mathcal{S}(B^c(\mathrm{Perm}^\vee), \mathcal{S}(B^c(\mathrm{Perm}^\vee), L)) \longrightarrow \mathcal{S}(B^c(\mathrm{Perm}^\vee), L)$ and $\widetilde{\mu} : \Gamma(B^c(\mathrm{Perm}^\vee), \Gamma(B^c(\mathrm{Perm}^\vee), L)) \longrightarrow \Gamma(B^c(\mathrm{Perm}^\vee), L)$ the monadic compositions. Let $x \in L$ and $B_1, \ldots, B_n \in \mathcal{S}^r(B^c(\mathrm{Perm}^\vee), L)$ be basis elements. Then*

$$\widetilde{\mu}(\mathcal{O}\Sigma^{-1}e_1^{n+1}(x, \mathcal{O}B_1, \ldots, \mathcal{O}B_n)) = \mathcal{O}(\mu(\Sigma^{-1}e_1^{n+1}(x, B_1, \ldots, B_n))).$$

Proof. The proof is identical to the proofs given in [Ces18, Theorem 1.5.1, Lemma 1.5.2]. □

Theorem 2.2.30. *A dg $\mathbb{K}$-module (L, d) is a $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebra if and only if $\Sigma L \in \Gamma\Lambda\mathcal{PL}_\infty$ with $Q_0^0 = \Sigma d$. Moreover, every morphism of $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebras $\phi : L \longrightarrow L'$ gives rise to a strict morphism $\Sigma\phi : \Sigma L \longrightarrow \Sigma L'$ in $\Gamma\Lambda\mathcal{PL}_\infty$.*

Proof. Let L be a $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebra. Then ΣL is a $\Gamma(\Lambda\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebra by Proposition 2.1.17. Since $\Lambda\mathcal{P}re\mathcal{L}ie_\infty \simeq B^c(\mathrm{Perm}^\vee)$, we have a morphism

$$l : \Gamma(\Sigma^{-1}\overline{\mathrm{Perm}}^\vee, \Sigma L) \longrightarrow \Sigma L.$$

We then set, for homogeneous elements $x, y_1, \dots, y_n \in \Sigma L$, and $r_1, \dots, r_n \geq 0$,

$$x\{\!\{y_1, \dots, y_n\}\!\}_{r_1,\dots,r_n} := l(\mathcal{O}(\Sigma^{-1}e_1^{r+1} \otimes x \otimes y_1^{\otimes r_1} \otimes \cdots \otimes y_n^{\otimes r_n})),$$

where $r = r_1 + \cdots + r_n$ and where the considered orbit map is using a basis which includes $x, y_1, \dots, y_n$. We check all formulas given in Theorem 2.2.22. Formulas $(i)-(v)$ come from straightforward computations. We prove formula (vi). We compute

$$d(x\{\!\{y_1, \dots, y_n\}\!\}_{r_1,\dots,r_n}) = ld(\mathcal{O}(\Sigma^{-1}e_1^{r+1} \otimes x \otimes y_1^{\otimes r_1} \otimes \cdots \otimes y_n^{\otimes r_n})).$$

We have

$$\mathcal{O}(\Sigma^{-1}e_1^{r+1} \otimes x \otimes y_1^{\otimes r_1} \otimes \cdots \otimes y_n^{\otimes r_n}) = \sum_{\sigma \in Sh(1,r_1,\dots,r_n)} \sigma \cdot (\Sigma^{-1}e_1^{r+1} \otimes x \otimes y_1^{\otimes r_1} \otimes \cdots \otimes y_n^{\otimes r_n}).$$

Let ∂ be the differential of $B^c(\mathrm{Perm}^\vee)$. Then

$$\begin{aligned} d(x\{\!\{y_1, \dots, y_n\}\!\}_{r_1,\dots,r_n}) = l &\left(\sum_{\sigma \in Sh(1,r_1,\dots,r_n)} \sigma \cdot (\partial(\Sigma^{-1}e_1^{r+1}) \otimes x \otimes y_1^{\otimes r_1} \otimes \cdots \otimes y_n^{\otimes r_n}) \right) \\ &+ d(x)\{\!\{y_1, \dots, y_n\}\!\}_{r_1,\dots,r_n} + \sum_{k=1}^n \pm x\{\!\{y_1, \dots, y_k, d(y_k), \dots, y_n\}\!\}_{r_1,\dots,r_k-1,1,\dots,r_n}. \end{aligned}$$

We compute the first sum. Recall from the operadic composition in Perm (see Proposition 2.2.1) and from the definition of the differential in the cobar construction of a coaugmented cooperad that we have

$$\partial(\Sigma^{-1}e_1^{r+1}) = \sum_{\substack{p+q=r+2 \\ p,q \geq 2}} \left(\sum_{\omega \in Sh_*(q,1,\dots,1)} \omega \cdot T_{p,q}^{1,1} + \sum_{k=2}^p \sum_{i=1}^q \sum_{\omega \in Sh_*(1,\dots,\underset{k}{q},\dots,1)} \omega \cdot T_{p,q}^{k,i} \right),$$

where we have set

$$T_{p,q}^{k,i} = \begin{array}{c} \qquad\qquad\qquad\quad k \ \cdots \ k+q-1 \\ \qquad\qquad\qquad\quad \diagdown \quad \diagup \\ 1 \ \cdots \ k-1 \ \Sigma^{-1}e_i^q \ k+q \ \cdots \ p+q-1 \\ \diagdown \quad \diagdown \ | \ \diagup \quad \diagup \\ \Sigma^{-1}e_1^p \end{array}.$$

Let $z_1, \ldots, z_{r+1} = x, \underbrace{y_1, \ldots, y_1}_{r_1}, \ldots, \underbrace{y_n, \ldots, y_n}_{r_n}$. For given $p, q \geq 2$ such that $p+q = r+2$, we need to compute the sums,

$$S^1_{p,q} := \sum_{\sigma \in Sh(1,r_1,\ldots,r_n)} \sum_{\omega \in Sh_*(q,1,\ldots,1)} \pm\sigma\omega \cdot (T^{1,1}_{p,q} \otimes z_{\omega(1)} \otimes \cdots \otimes z_{\omega(r+1)});$$

$$S^2_{p,q} := \sum_{k=2}^{p} \sum_{i=1}^{q} \sum_{\sigma \in Sh(1,r_1,\ldots,r_n)} \sum_{\omega \in Sh_*(1,\ldots,\underset{k}{q},\ldots,1)} \pm\sigma\omega \cdot (T^{k,i}_{p,q} \otimes z_{\omega(1)} \otimes \cdots \otimes z_{\omega(r+1)}).$$

We first compute $S^1_{p,q}$. We claim that

$$S^1_{p,q} = \sum_{\substack{p_i+q_i=r_i \\ p_1+\cdots+p_n=p-1 \\ q_1+\cdots+q_n=q-1}} \pm\mathcal{O}(T^{1,1}_{p,q} \otimes x \otimes y_1^{\otimes q_1} \otimes \cdots \otimes y_n^{\otimes q_n} \otimes y_1^{\otimes p_1} \otimes \cdots \otimes y_n^{\otimes p_n}).$$

Let $\sigma \in Sh(r_1, \ldots, r_n)$ and $\omega \in Sh_*(q, 1, \ldots, 1)$. In particular we have $\omega \in Sh(1, q-1, p-1)$ with $\omega(1) = 1$. Then there exist $p_1, \ldots, p_n, q_1, \ldots, q_n$ with $p_i + q_i = r_i, p_1 + \cdots + p_n = p-1$ and $q_1 + \cdots + q_n = q-1$ such that

$$z_{\omega(1)} \otimes \cdots \otimes z_{\omega(r+1)} = \pm x \otimes y_1^{\otimes q_1} \otimes \cdots \otimes y_n^{\otimes q_n} \otimes y_1^{\otimes p_1} \otimes \cdots \otimes y_n^{\otimes p_n}.$$

Thus, every term in the left hand-side sum is part of the sum in the right hand-side. We now consider an element which occurs in the expansion of $\mathcal{O}(T^{1,1}_{p,q} \otimes x \otimes y_1^{\otimes q_1} \otimes \cdots \otimes y_n^{\otimes q_n} \otimes y_1^{\otimes p_1} \otimes \cdots \otimes y_n^{\otimes p_n})$ for some p_i's,q_i's as above. Let $\beta \in \Sigma_{r+1}$ be a blocks permutation which sends $x \otimes y_1^{\otimes q_1} \otimes \cdots \otimes y_n^{\otimes q_n} \otimes y_1^{\otimes p_1} \otimes \cdots \otimes y_n^{\otimes p_n}$ to $\pm x \otimes y_1^{\otimes r_1} \otimes \cdots \otimes y_n^{\otimes r_n}$. Then

$$\mathcal{O}(T^{1,1}_{p,q} \otimes x \otimes y_1^{\otimes q_1} \otimes \cdots \otimes y_n^{\otimes q_n} \otimes y_1^{\otimes p_1} \otimes \cdots \otimes y_n^{\otimes p_n}) = \pm\mathcal{O}(\beta \cdot T^{1,1}_{p,q} \otimes x \otimes y_1^{\otimes r_1} \otimes \cdots \otimes y_n^{\otimes r_n}).$$

Let $\tau \in \Sigma_{r+1}$. We write $\tau = \sigma\eta$ where $\sigma \in Sh(1, r_1, \ldots, r_n)$ and $\eta \in \Sigma_1 \times \Sigma_{r_1} \times \cdots \times \Sigma_{r_n}$. Then

$$\tau \cdot (\mu \cdot T^{1,1}_{p,q} \otimes x \otimes y_1^{\otimes r_1} \otimes \cdots \otimes y_n^{\otimes r_n}) = \sigma \cdot (\eta\beta \cdot T^{1,1}_{p,q} \otimes x \otimes y_1^{\otimes r_1} \otimes \cdots \otimes y_n^{\otimes r_n}).$$

We write $\eta\beta = \omega\nu$ where $\omega \in Sh(1, q-1, p-1)$ and $\nu \in \Sigma_1 \times \Sigma_{q-1} \times \Sigma_{p-1}$. Since $\eta\beta(1) = 1$, we have $\omega(1) = 1$ so that $\omega \in Sh_*(q, 1, \ldots, 1)$. We finally have

$$\tau \cdot (\mu \cdot T^{1,1}_{p,q} \otimes x \otimes y_1^{\otimes r_1} \otimes \cdots \otimes y_n^{\otimes r_n}) = \sigma \cdot (\omega \cdot T^{1,1}_{p,q} \otimes x \otimes y_1^{\otimes r_1} \otimes \cdots \otimes y_n^{\otimes r_n})$$

so that every term in the right hand-side is part of $S^1_{p,q}$. We thus have proved the first identity.

We now compute $S^2_{p,q}$. We claim that

$$S^2_{p,q} = \sum_{j=1}^{n} \sum_{\substack{p_i+q_i=r_i, i\neq j\\ p_j+q_j=r_j-1\\ p_1+\cdots+p_n=p-1\\ q_1+\cdots+q_n=q-1}} \pm\mathcal{O}(T^{2,1}_{p,q}\otimes x\otimes y_j\otimes y_1^{\otimes q_1}\otimes\cdots\otimes y_n^{\otimes q_n}\otimes y_1^{\otimes p_1}\otimes\cdots\otimes y_n^{\otimes p_n}).$$

Since there exists $\nu\in\Sigma_{r+1}$ such that $T^{k,i}_{p,q}=\nu\cdot T^{2,1}_{p,q}$, we can apply the same arguments as before to show that every term which occurs in $S^2_{p,q}$ is part of the right-hand side sum. Now consider some term $T^{2,1}_{p,q}\otimes x\otimes y_j\otimes y_1^{\otimes q_1}\otimes\cdots\otimes y_n^{\otimes q_n}\otimes y_1^{\otimes p_1}\otimes\cdots\otimes y_n^{\otimes p_n}$. Let $\beta\in\Sigma_{r+1}$ be a blocks permutation which sends $x\otimes y_j\otimes y_1^{\otimes q_1}\otimes\cdots\otimes y_n^{\otimes q_n}\otimes y_1^{\otimes p_1}\otimes\cdots\otimes y_n^{\otimes p_n}$ to $\pm x\otimes y_1^{\otimes r_1}\otimes\cdots\otimes y_n^{\otimes r_n}$. Then

$$\mathcal{O}(T^{2,1}_{p,q}\otimes x\otimes y_j\otimes y_1^{\otimes q_1}\otimes\cdots\otimes y_n^{\otimes q_n}\otimes y_1^{\otimes p_1}\otimes\cdots\otimes y_n^{\otimes p_n}) = \pm\mathcal{O}(\beta\cdot T^{2,1}_{p,q}\otimes x\otimes y_1^{\otimes r_1}\otimes\cdots\otimes y_n^{\otimes r_n}).$$

Now let $\tau\in\Sigma_{r+1}$. We write $\tau=\sigma\eta$ where $\sigma\in Sh(1,r_1,\ldots,r_n)$ and $\eta\in\Sigma_1\times\Sigma_{r_1}\times\cdots\times\Sigma_{r_n}$. Then

$$\tau\cdot(\beta\cdot T^{2,1}_{p,q}\otimes x\otimes y_1^{\otimes r_1}\otimes\cdots\otimes y_n^{\otimes r_n}) = \sigma\cdot(\eta\beta\cdot T^{2,1}_{p,q}\otimes x\otimes y_1^{\otimes r_1}\otimes\cdots\otimes y_n^{\otimes r_n}).$$

We finally write $\eta\beta=\omega\cdot\nu_*(1,\gamma,1,\ldots,1)$ where $\nu\in\Sigma_p,\gamma\in\Sigma_q,\omega\in Sh_*(1,\ldots,\underset{k}{q},\ldots,1)$ and $k=\nu(2)$. Since $\eta\beta(1)=1$, we have $\nu(1)=1$. We thus obtain

$$\tau\cdot(\beta\cdot T^{2,1}_{p,q}\otimes x\otimes y_1^{\otimes r_1}\otimes\cdots\otimes y_n^{\otimes r_n}) = \sigma\cdot(\omega\cdot T^{k,\gamma(1)}_{p,q}\otimes x\otimes y_1^{\otimes r_1}\otimes\cdots\otimes y_n^{\otimes r_n})$$

so that every term in the right-hand side is part of $S^2_{p,q}$.

From Lemma 2.2.29, we deduce that formula (vi) of Theorem 2.2.22 is satisfied so that $\Sigma L\in\Gamma\Lambda\mathcal{PL}_\infty$. Suppose now that L is such that $\Sigma L\in\Gamma\Lambda\mathcal{PL}_\infty$. We prove that L is a $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty,-)$-algebra, or equivalently, that ΣL is a $\Gamma(B^c(\mathrm{Perm}^\vee),-)$-algebra (see Proposition 2.1.17). We first define

$$l:\bigoplus_{n\geq 0}(\Sigma^{-1}\overline{\mathrm{Perm}^\vee}(n)\otimes(\Sigma L)^{\otimes n})^{\Sigma_n}\longrightarrow\Sigma L$$

by setting, for every basis elements $x,y_1,\ldots,y_n\in\Sigma L$,

$$l(\mathcal{O}(\Sigma^{-1}e_1^{\sum_i r_i+1}\otimes x\otimes y_1^{\otimes r_1}\otimes\cdots\otimes y_n^{\otimes r_n})) := x\{\!\{y_1,\ldots,y_n\}\!\}_{r_1,\ldots,r_n}.$$

We then extend $l:\Gamma(B^c(\mathrm{Perm}^\vee),\Sigma L)\longrightarrow\Sigma L$ by Lemma 2.2.29. By the same identities as before, we can show that l preserves the differentials, giving a structure of a $\Gamma(\Lambda\mathcal{P}re\mathcal{L}ie_\infty,-)$-algebra on ΣL. □

Remark 2.2.31. *This theorem implies that the category of the $\Gamma(\Lambda\mathcal{P}re\mathcal{L}ie_\infty,-)$-algebras is a full subcategory of $\Gamma\Lambda\mathcal{PL}_\infty$. However, a morphism in $\Gamma\Lambda\mathcal{PL}_\infty$ does not necessarily preserves the monadic structure of $\Gamma(\Lambda\mathcal{P}re\mathcal{L}ie_\infty,-)$.*

Corollary 2.2.32. *For every complete $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra L, the dg $\mathbb{K}$-module ΣL is endowed with the structure of a $\widehat{\Gamma\Lambda\mathcal{PL}_\infty}$-algebra such that $\mathcal{MC}(L)$ is in bijective correspondence with $\mathcal{MC}(\Sigma L)$.*

Proof. The operad morphism $\mathcal{P}re\mathcal{L}ie_\infty \longrightarrow \mathcal{P}re\mathcal{L}ie$ given in Remark 2.2.7 gives rise to a monad morphism $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -) \longrightarrow \Gamma(\mathcal{P}re\mathcal{L}ie, -)$. Then, every $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra L is endowed with the structure of a $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebra. By Theorem 2.2.30, the dg $\mathbb{K}$-module ΣL is a $\Gamma\Lambda\mathcal{PL}_\infty$-algebra. If we denote by $-\{-, \ldots, -\}_{r_1,\ldots,r_n}$ the weighted brace operations given by the $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra structure on L, then, by definition of the weighted brace operations given in the proof of Theorem 2.2.30, we have

$$\Sigma x\{\!\{\Sigma y_1, \ldots, \Sigma y_n\}\!\}_{r_1,\ldots,r_n} = \begin{cases} -\Sigma d(x) & \text{if } r_1 + \cdots + r_n = 0 \\ (-1)^{|x|}\Sigma x\{y_1\}_1 & \text{if } r_1 = 1, r_2 = \ldots = r_n = 0 \\ 0 & \text{if } r_1 + \cdots + r_n \geq 2 \end{cases}$$

for every $x, y_1, \ldots, y_n \in L$. Since the operations $-\{-, \ldots, -\}_{r_1,\ldots,r_n}$ preserve the filtration on L, the operations $-\{\!\{-, \ldots, -\}\!\}_{r_1,\ldots,r_n}$ also preserve the filtration on ΣL so that $\Sigma L \in \widehat{\Gamma\Lambda\mathcal{PL}_\infty}$. Moreover, the previous computation of the braces shows that $x \in \mathcal{MC}(L)$ if and only if $\Sigma x \in \mathcal{MC}(\Sigma L)$, which proves the corollary. $\square$

Corollary 2.2.33. *Suppose that $char(\mathbb{K}) = 0$. Then every $\Gamma\Lambda\mathcal{PL}_\infty$-algebra V is endowed with the structure of a $\Lambda\mathcal{PL}_\infty$-algebra such that*

$$x\{\!\{y_1, \ldots, y_n\}\!\}_{r_1,\ldots,r_n} = \frac{1}{\prod_i r_i!} x\{\!\{\underbrace{y_1, \ldots, y_1}_{r_1}, \ldots, \underbrace{y_n, \ldots, y_n}_{r_n}\}\!\}$$

for every $x \in V$ and $y_1, \ldots, y_n \in V$ with associated weights $r_1, \ldots, r_n \geq 0$, where we consider the symmetric braces $-\{\!\{-, \ldots, -\}\!\}$ defined in Proposition 2.2.6.

Proof. Let V be a $\Gamma\Lambda\mathcal{PL}_\infty$-algebra. By Theorem 2.2.30, we have that $\Sigma^{-1}V$ is a $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebra. Since we have a morphism of monads given by the trace map

$$Tr : \mathcal{S}(\mathcal{P}re\mathcal{L}ie_\infty, -) \longrightarrow \Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -),$$

$\Sigma^{-1}V$ is endowed with a $\mathcal{P}re\mathcal{L}ie_\infty$-algebra structure, so that V is a $\Lambda\mathcal{PL}_\infty$-algebra. By using the definition of weighted brace operations, we obtain the desired relation. $\square$

2.3 A morphism from $\mathcal{P}re\mathcal{L}ie_\infty$ to $\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$

In this section, we construct an operad morphism from $\mathcal{P}re\mathcal{L}ie_\infty$ to $\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$. We will define this morphism as a composite of the form $\mathcal{P}re\mathcal{L}ie_\infty \longrightarrow \mathcal{B}race \underset{\mathrm{H}}{\otimes} B^c(\Lambda^{-1}\mathcal{B}race^\vee) \longrightarrow \mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$.

In §2.3.1, we construct an operad morphism $B^c(\Lambda^{-1}\mathcal{B}race^\vee) \longrightarrow \mathcal{E}$, which will be given as a lift of some diagram.

In §2.3.2, we give a computation of the twisted coderivation on $\mathcal{B}race^c(\Sigma^{-1}N_*(\Delta^n))$ induced by the morphism constructed in §2.3.1 and the $\mathcal{E}$-coalgebra structure on $N_*(\Delta^n)$ by induction on $n \geq 0$.

In §2.3.3, we construct the morphism $\mathcal{P}re\mathcal{L}ie_\infty \longrightarrow \mathcal{B}race \underset{\mathrm{H}}{\otimes} B^c(\Lambda^{-1}\mathcal{B}race^\vee)$ and deduce Theorem D.

2.3.1 A morphism from $B^c(\Lambda^{-1}\mathcal{B}race^\vee)$ to $\mathcal{E}$

Let $\mathcal{C}om$ be the commutative operad. Recall that if we consider the model structure on the category of symmetric operads $\mathcal{P}$ such that $\mathcal{P}(0) = 0$ (see [Hin03, §3.3]), then we have an acyclic fibration $\mathcal{E} \overset{\sim}{\twoheadrightarrow} \mathcal{C}om$. We thus have there exists a lift of the following diagram:

$$\begin{tikzcd} & \mathcal{E} \arrow[d, two heads, "\sim"] \\ B^c(\Lambda^{-1}\mathcal{B}race^\vee) \arrow[r] \arrow[ur, dashed, "\exists"] & \mathcal{C}om \end{tikzcd} .$$

The goal of this subsection is to give an explicit choice of such a lift. Equivalently, we are searching for elements $\mu_T \in \mathcal{E}(|T|)_{|T|-2}$ for every $T \in \mathcal{PRT}$ such that

$$d(\Lambda\mu_T) + \sum_{S \subset T} \Lambda\mu_{T/S} \circ_S \Lambda\mu_S = 0.$$

For every $\sigma \in \Sigma_r$, let $h^\sigma_{\mathcal{E}} : \mathcal{E}(r)_d \longrightarrow \mathcal{E}(r)_{d+1}$ be the morphism such that, for every $w_0, \ldots, w_d \in \Sigma_r$,

$$h^\sigma_{\mathcal{E}}(w_0, \ldots, w_r) = (\sigma, w_0, \ldots, w_r).$$

We can check that this morphism is a homotopy between the identity map of $\mathcal{E}(r)_d$ and the morphism $\varphi^\sigma_{\mathcal{E}} : \mathcal{E}(r)_d \longrightarrow \mathcal{E}(r)_d$ defined by

$$\varphi^\sigma_{\mathcal{E}}(w_0, \ldots, w_d) = \begin{cases} \sigma & \text{if } d = 0 \\ 0 & \text{else} \end{cases} .$$

Accordingly,

$$dh^\sigma_{\mathcal{E}} + h^\sigma_{\mathcal{E}} d = id_{\mathcal{E}} - \varphi^\sigma_{\mathcal{E}}.$$

By functoriality, we have that $\Lambda h^\sigma_{\mathcal{E}}$ is a homotopy between the identity map and $\Lambda\varphi^\sigma_{\mathcal{E}}$.

Construction 2.3.1. *We set* $\mu_{1} = 0$, $\mu_{\substack{2\\|\\1}} = (12)$ *and* $\mu_{\substack{1\\|\\2}} = (21)$. *For every* $T \in \mathcal{PRT}$, *we define* μ_T *by induction on* $|T|$ *by*

$$\Lambda\mu_T = -\Lambda h^{\sigma_T}_{\mathcal{E}} \left(\sum_{S \subset T} \Lambda\mu_{T/S} \circ_S \Lambda\mu_S \right),$$

where we take $\mu_{T/S} \in \mathcal{E}(V_{T/S})$ *and* $\mu_S \in \mathcal{E}(V_S)$ *(see Remark 2.1.10).*

Example 2.3.2. *Let us make explicit the* μ_T*'s for* T *a tree with* 3 *vertices.*

— *If* $T = \substack{2\ \ 3\\ \vee\\ 1}$, *then we have two non trivial trees* $S_1 = \substack{2\\|\\1}$ *and* $S_2 = \substack{3\\|\\1}$, *which are such that* $T/S_1 = \substack{3\\|\\S_1}$ *and* $T/S_2 = \substack{2\\|\\S_2}$. *We thus have*

$$\begin{aligned} \sum_{S\subset T} \Lambda\mu_{T/S} \circ_S \Lambda\mu_S &= \Sigma^{-1}(S_1 3) \circ_{S_1} \Sigma^{-1}(12) + \Sigma^{-1}(S_2 2) \circ_{S_2} \Sigma^{-1}(13) \\ &= \Sigma^{-1}(123) - \Sigma^{-1}(132). \end{aligned}$$

We then deduce $\Lambda\mu_T = \Sigma^{-1}(123, 132)$.

— *If* $T = \begin{array}{c} 3 \\ | \\ 2 \\ | \\ 1 \end{array}$, *then we have two non trivial trees* $S_1 = \begin{array}{c} 2 \\ | \\ 1 \end{array}$ *and* $S_2 = \begin{array}{c} 3 \\ | \\ 2 \end{array}$, *which are such that* $T/S_1 = \begin{array}{c} 3 \\ | \\ S_1 \end{array}$ *and* $T/S_2 = \begin{array}{c} S_2 \\ | \\ 1 \end{array}$. *We thus have*

$$\begin{aligned} \sum_{S\subset T} \Lambda\mu_{T/S} \circ_S \Lambda\mu_S &= \Sigma^{-1}(S_1 3) \circ_{S_1} \Sigma^{-1}(12) + \Sigma^{-1}(1 S_2) \circ_{S_2} \Sigma^{-1}(23) \\ &= \Sigma^{-1}(123) - \Sigma^{-1}(123) \\ &= 0. \end{aligned}$$

We then deduce $\Lambda\mu_T = 0$.

Theorem 2.3.3. *For every tree* $T \in \mathcal{PRT}$, *we have*

$$d(\Lambda\mu_T) + \sum_{S\subset T} \Lambda\mu_{T/S} \circ_S \Lambda\mu_S = 0$$

where we consider the elements μ_T*'s defined in Construction 2.3.1. Then we have an explicit lift*

$$\begin{tikzcd} & \mathcal{E} \arrow[d, two heads, "\sim"] \\ B^c(\Lambda^{-1}\mathcal{B}race^\vee) \arrow[r] \arrow[ur, dashed] & \mathcal{C}om \end{tikzcd}$$

given by the morphism

$$\begin{array}{ccc} B^c(\Lambda^{-1}\mathcal{B}race^\vee) & \longrightarrow & \mathcal{E} \\ \Sigma^{-1}\Lambda^{-1}T^\vee & \longmapsto & \mu_T \end{array}.$$

Proof. The theorem obviously holds if $|T| = 1, 2$, and also if $|T| = 3$ by Example 2.3.2. We now suppose that $n := |T| \geq 4$. We use the identity $d\Lambda h_{\mathcal{E}}^{\sigma_T} + \Lambda h_{\mathcal{E}}^{\sigma_T} d = id_{\Lambda\mathcal{E}} - \Lambda\varphi_{\mathcal{E}}^{\sigma_T}$:

$$d(\Lambda\mu_T) = \Lambda h_{\mathcal{E}}^{\sigma_T} d \left(\sum_{S\subset T} \Lambda\mu_{T/S} \circ_S \Lambda\mu_S \right) + \Lambda\varphi_{\mathcal{E}}^{\sigma_T} \left(\sum_{S\subset T} \Lambda\mu_{T/S} \circ_S \Lambda\mu_S \right) - \sum_{S\subset T} \Lambda\mu_{T/S} \circ_S \Lambda\mu_S.$$

By an immediate induction, for every non trivial tree T, we have that $\Lambda\mu_T \in \Lambda\mathcal{E}(n)_{-1}$, or equivalently $\mu_T \in \mathcal{E}(n)_{n-2}$. This then gives $|\mu_{T/S} \circ_S \mu_S| = n - 3 > 0$. Thus, by definition of $\varphi_{\mathcal{E}}^{\sigma_T}$, we have that

$$\Lambda\varphi_{\mathcal{E}}^{\sigma_T} \left(\sum_{S\subset T} \Lambda\mu_{T/S} \circ_S \Lambda\mu_S \right) = 0.$$

We now prove that

$$d \left(\sum_{S\subset T} \Lambda\mu_{T/S} \circ_S \Lambda\mu_S \right) = 0.$$

We compute:

$$d\left(\sum_{S\subset T}\Lambda\mu_{T/S}\circ_S\Lambda\mu_S\right)=\sum_{S\subset T}d(\Lambda\mu_{T/S})\circ_S\Lambda\mu_S-\sum_{S\subset T}\Lambda\mu_{T/S}\circ_S d(\Lambda\mu_S),$$

because the differential of the Barratt-Eccles operad is compatible with its operad structure. Now, because $\mu_S=0$ if S has only one vertex, we can consider subtrees of T with at most $n-1$ vertices. We can then use the induction hypothesis.

$$\begin{aligned}d\left(\sum_{S\subset T}\Lambda\mu_{T/S}\circ_S\Lambda\mu_S\right)=&-\sum_{S\subset T}\sum_{U\subset T/S}(\Lambda\mu_{(T/S)/U}\circ_U\Lambda\mu_U)\circ_S\Lambda\mu_S\\&+\sum_{S\subset T}\sum_{U\subset S}\Lambda\mu_{T/S}\circ_S(\Lambda\mu_{S/U}\circ_U\Lambda\mu_U)\end{aligned}$$

We have two types of subtrees U of T/S: either U does not contain the vertex S, so that U can be canonicaly seen as a subtree of T such that $V_U\cap V_S=\emptyset$, or U contains the vertex S, so that U can be seen as a subtree of T such that $S\subset U$. We thus have

$$\begin{aligned}d\left(\sum_{S\subset T}\Lambda\mu_{T/S}\circ_S\Lambda\mu_S\right)=&-\sum_{S\subset T}\sum_{S\subset U\subset T}(\Lambda\mu_{T/U}\circ_{U/S}\Lambda\mu_{U/S})\circ_S\Lambda\mu_S\\&-\sum_{\substack{S,U\subset T\\ V_S\cap V_U=\emptyset}}(\Lambda\mu_{(T/S)/U}\circ_U\Lambda\mu_U)\circ_S\Lambda\mu_S\\&+\sum_{S\subset T}\sum_{U\subset S}\Lambda\mu_{T/S}\circ_S(\Lambda\mu_{S/U}\circ_U\Lambda\mu_U).\end{aligned}$$

In the second line, by exchanging the roles of S and U, we have a sum of terms of the form

$$(\Lambda\mu_{(T/S)/U}\circ_U\Lambda\mu_U)\circ_S\Lambda\mu_S+(\Lambda\mu_{(T/U)/S}\circ_S\Lambda\mu_S)\circ_U\Lambda\mu_U$$

which is 0, because of the operadic axioms and because $|\Lambda\mu_U|=|\Lambda\mu_S|=-1$. We thus obtain

$$\begin{aligned}d\left(\sum_{S\subset T}\Lambda\mu_{T/S}\circ_S\Lambda\mu_S\right)&=-\sum_{U\subset T}\sum_{S\subset U}(\Lambda\mu_{(T/U)}\circ_{U/S}\Lambda\mu_{(U/S)})\circ_S\Lambda\mu_S\\&\quad+\sum_{S\subset T}\sum_{U\subset S}\Lambda\mu_{T/S}\circ_S(\Lambda\mu_{(S/U)}\circ_U\Lambda\mu_U)\\&=-\sum_{S\subset T}\sum_{U\subset S}(\Lambda\mu_{T/S}\circ_{S/U}\Lambda\mu_{(S/U)})\circ_U\Lambda\mu_U\\&\quad+\sum_{S\subset T}\sum_{U\subset S}\Lambda\mu_{T/S}\circ_S(\Lambda\mu_{(S/U)}\circ_U\Lambda\mu_U)\\&=0,\end{aligned}$$

using again the associativity of the operadic composition. □

2.3.2 On the twisted coderivation of $\mathcal{B}race^c(\Sigma N^*(\Delta^n))$

Every $\mathcal{E}$-algebra E inherits a $B^c(\Lambda^{-1}\mathcal{B}race^\vee)$-algebra structure induced by the morphism $B^c(\Lambda^{-1}\mathcal{B}race^\vee) \longrightarrow \mathcal{E}$ given by Theorem 2.3.3. This algebraic structure is equivalent to giving a twisted coderivation on the free brace coalgebra $\mathcal{B}race^c(\Sigma E)$ generated by ΣE. Recall that

$$\mathcal{B}race^c(\Sigma E) = \bigoplus_{k\geq 1} \bigoplus_{T \in \mathcal{PRT}(k)} (T^\vee \otimes (\Sigma E)^{\otimes k})^{\Sigma_k},$$

where we equalize the action of Σ_k on $T^\vee$ with the action of Σ_k by permutation on $(\Sigma E)^{\otimes k}$. In the following, we identify the $k=1$ component with ΣE.
If E is finite dimensional, giving such a coderivation is equivalent to giving a twisting morphism ∂^E on the free complete brace algebra

$$\widehat{\mathcal{B}race}(\Sigma^{-1}E^\vee) = \prod_{k\geq 1} \bigoplus_{T\in\mathcal{PRT}(k)} (T \otimes (\Sigma^{-1}E^\vee)^{\otimes k})_{\Sigma_k}.$$

This completion is obtained from the free brace algebra $\mathcal{B}race(\Sigma^{-1}E^\vee)$ endowed with the filtration

$$F_p\mathcal{B}race(\Sigma^{-1}E^\vee) = \bigoplus_{k\geq p+1} \bigoplus_{T\in\mathcal{PRT}(k)} (T \otimes (\Sigma^{-1}E^\vee)^{\otimes k})_{\Sigma_k}.$$

One can check that the brace algebra structure of $\mathcal{B}race(\Sigma^{-1}E^\vee)$ preserves this filtration so that the completion $\widehat{\mathcal{B}race}(\Sigma^{-1}E^\vee)$ is endowed with a brace algebra structure. The derivation ∂^E is thus given by a generating function

$$\partial^E = \sum_{\substack{T\in\mathcal{PRT} \\ T \text{ canonical}}} T \otimes \Lambda\mu_T^{E^\vee}$$

where $\mu_T^{E^\vee} : E^\vee \longrightarrow (E^\vee)^{\otimes|T|}$ is the map induced by $\mu_T \in \mathcal{E}(|T|)$ given by the $\mathcal{E}$-algebra structure of E. Note also that the definition of ∂^E is natural with respect to E. Namely, if F is an other finite dimensional $\mathcal{E}$-algebra and $f : F \longrightarrow E$ a morphism of $\mathcal{E}$-algebras, then we have a commutative diagram

$$\begin{tikzcd}
\widehat{\mathcal{B}race}(\Sigma^{-1}E^\vee) \arrow[r, "\widehat{\mathcal{B}race}(\Sigma^{-1}f^\vee)"] \arrow[d, "\partial^E"'] & \widehat{\mathcal{B}race}(\Sigma^{-1}F^\vee) \arrow[d, "\partial^F"] \\
\widehat{\mathcal{B}race}(\Sigma^{-1}E^\vee) \arrow[r, "\widehat{\mathcal{B}race}(\Sigma^{-1}f^\vee)"'] & \widehat{\mathcal{B}race}(\Sigma^{-1}F^\vee)
\end{tikzcd}.$$

The goal of this subsection is to give a computation of the differentials $\partial^n := \partial^{N^*(\Delta^n)}$ by induction on $n \geq 0$. To achieve this, we use Construction 2.3.1 which defines the $\Lambda\mu_T$'s, and analyze the coaction of $TR(\Lambda\mu_T)$ on $\Sigma^{-1}\underline{0\cdots n} \in \Sigma^{-1}N_*(\Delta^n)$, where TR is the table reduction morphism (see Proposition 2.1.29).

Let $r, d \geq 0$ and $1 \leq k \leq r$. We denote by

$$\pi_k : \chi(r)_d \longrightarrow \chi(\llbracket k, r \rrbracket)_d$$

the morphism obtained by forgetting $1, \ldots, k-1$. If the degree does not match, we send the surjection on 0. Note that $\pi_1 = id$.

Lemma 2.3.4. *Let $w \in \mathcal{E}(r)_d$. Then*

$$TR(h_{\mathcal{E}}^{id}(w)) = \sum_{k=1}^{r} 1 \cdots k \cdot \pi_k(TR(w)),$$

where we consider the concatenation of a surjection in $\chi(k)$ with a surjection of $\chi(\llbracket k, r \rrbracket)$, giving a surjection in $\chi(r)$.

Proof. Let $w = (w_0, \ldots, w_d) \in \mathcal{E}(r)_d$. On one hand, we have

$$TR(h_{\mathcal{E}}^{id}(w)) = \sum_{k=1}^{r} \sum_{\substack{r_0 + \cdots + r_d = r+d-k+1 \\ 1 \leq r_i \leq r}} \left| \begin{array}{ccc} 1 & \cdots & k \\ w'_0(1) & \cdots & w'_0(r_0) \\ \vdots & & \vdots \\ w'_d(1) & \cdots & w'_d(r_d) \end{array} \right.$$

where each $w'_k(1), \ldots, w'_k(r_k)$ are obtained from $w_k(1), \ldots, w_k(r)$ by taking the first r_k terms which are not among

$$\begin{array}{ccc} 1 & \cdots & k-1 \\ w'_0(1) & \cdots & w'_0(r_0 - 1) \\ \vdots & & \vdots \\ w'_{k-1}(1) & \cdots & w'_{k-1}(r_{k-1} - 1) \end{array}.$$

On the other hand, if we write

$$TR(w) = \sum_{\substack{r'_0 + \cdots + r'_d = r+d \\ 1 \leq r_i \leq r}} \left| \begin{array}{ccc} w''_0(1) & \ldots & w''_0(r'_0) \\ \vdots & & \vdots \\ w''_d(1) & \ldots & w''_d(r'_d) \end{array} \right.$$

where each $w''_k(1), \ldots, w''_k(r'_k)$ are obtained from $w_k(1), \ldots, w_k(r)$ by taking the first r'_k terms which are not among

$$\begin{array}{ccc} w''_0(1) & \cdots & w''_0(r'_0 - 1) \\ \vdots & & \vdots \\ w''_{k-1}(1) & \cdots & w''_{k-1}(r'_{k-1} - 1) \end{array},$$

then

$$\pi_k(TR(w)) = \sum_{\substack{r_0 + \cdots + r_d = r+d-k+1 \\ 1 \leq r_i \leq r}} \left| \begin{array}{ccc} w'_0(1) & \ldots & w'_0(r_0) \\ \vdots & & \vdots \\ w'_d(1) & \ldots & w'_d(r_d) \end{array} \right.$$

where, as above, each $w'_k(1), \ldots, w'_k(r_k)$ are obtained from $w_k(1), \ldots, w_k(r)$ by taking the first r_k terms which are not among

$$\begin{matrix} 1 & \cdots & k-1 \\ w_0'(1) & \cdots & w_0'(r_0-1) \\ \vdots & & \vdots \\ w_{k-1}'(1) & \cdots & w_{k-1}'(r_{k-1}-1) \end{matrix}.$$

We then deduce

$$\sum_{k=1}^{r} 1\cdots k \cdot \pi_k(TR(w)) = \sum_{k=1}^{r} \sum_{\substack{r_0+\cdots+r_d=r+d-k+1 \\ 1\le r_i\le r}} \left| \begin{matrix} 1 & \cdots & k \\ w_0'(1) & \cdots & w_0'(r_0) \\ \vdots & & \vdots \\ w_d'(1) & \cdots & w_d'(r_d) \end{matrix} \right.$$

which proves the lemma. □

Lemma 2.3.5. *Let X be a totally ordered finite set and $T \in \mathcal{PRT}(X)$ with $|T| \geq 3$. We let b_T to be the number of vertices in the first branch of T (without the root).*

— *If $b_T \geq 2$, then $TR(\mu_T) = 0$.*

— *If $b_T = 1$, then, if we denote by r the root of T and s the second element of V_T, then there exists $u_T \in \chi(X \setminus \{r\})$ such that $TR(\mu_T) = rs \cdot u_T$.*

Proof. It is sufficient to prove the lemma for a canonical tree $T \in \chi(1 < \cdots < |T|)$. We prove the lemma by induction on $|T|$. If $|T| = 3$, then Example 2.3.2 implies that that the assertion of the lemma is true. We now suppose that $|T| \geq 4$. Suppose first that $b_T \geq 2$. By Lemma 2.3.4,

$$TR(\mu_T) = -\sum_{k=2}^{|T|} \sum_{S\subset T} \pm 1\cdots k \cdot \pi_k(TR(\mu_{T/S}) \circ_S TR(\mu_S)).$$

Note that the sum begins at $k = 2$, since that, by an immediate induction, the elements $TR(\mu_T)$'s begins at 1. Let $S \subset T$. Our goal is to prove that, for every $2 \leq k \leq |T|$,

$$1\cdots k \cdot \pi_k(TR(\mu_{T/S}) \circ_S TR(\mu_S)) = 0$$

We distinguish several cases. If either $b_S \geq 2$ or $b_{T/S} \geq 2$, then the identity holds by induction hypothesis. Suppose now that $b_S = b_{T/S} = 1$. Since $b_T \geq 2$, we have these different cases.

— If $r(S)$ is one of the vertex of the first branch of T, then, because $b_{T/S} = 1$, the tree S is the full first branch of T. In this situation, the first permutation occurring in $\mu_{T/S} \circ_S \mu_S$ is the identity permutation, so that taking the image of this element under $h_{\mathcal{E}}^{id}$ gives 0.

— If $r(S) = 1$, since $b_S = b_{T/S} = 1$, we have $b_T = 2$, and the second vertex of S is 2. Then, by induction hypothesis,

$$TR(\mu_S) = 12 \cdot u_S;$$

$$TR(\mu_{T/S}) = S3 \cdot u_{T/S},$$

where $u_S \in \chi(V_S \setminus \{1\})$ and $u_{T/S} \in \chi(V_{T/S} \setminus \{S\})$. We thus have

$$TR(\mu_{T/S}) \circ_S TR(\mu_S) = 12 \cdot u_S \cdot 3 \cdot u_{T/S}.$$

Since 2 occurs in the surjection u_S, taking the image of such an element under π_k will gives 0 for every $k \geq 3$. If $k = 2$, then the corresponding term is

$$12 \cdot \pi_2(12 \cdot u_S \cdot 3 \cdot u_{T/S}) = 122 \cdot u_S \cdot 3 \cdot u_{T/S} = 0.$$

— If $r(S)$ is neither 1 nor an element of the first branch of T, then we cannot have $b_{T/S} = 1$, since we have supposed $b_T \geq 2$.

This concludes the case $b_T \geq 2$. We now suppose that $b_T = 1$. We use again the identity up to signs

$$TR(\mu_T) = -\sum_{k=2}^{|T|} \sum_{S \subset T} \pm 1 \cdots k \cdot \pi_k(TR(\mu_{T/S}) \circ_S TR(\mu_S)),$$

given by Lemma 2.3.4. Our goal is to prove that the terms of this sum with $k \geq 3$ are 0. Let $S \subset T$ be such that $b_S = b_{T/S} = 1$. We have three cases.

— If $|S|, |T/S| \neq 2$, then, since $b_T = 1$, we cannot have $r(S) = 2$ so that 2 is the second occurrence of either the surjection $TR(\mu_S)$ or the surjection $TR(\mu_{T/S})$. By induction hypothesis, in any case, taking the image of π_k for every $k \geq 3$ of the corresponding element gives 0.

— Suppose now that $|S| = 2$ and $|T/S| \neq 2$. If S does not contain 2, then the above argument gives a resulting element equal to 0 in the sum, for $k \geq 3$. Else, we have $r(S) = 1$ so that $TR(\mu_S) = 12$. We thus have, by induction hypothesis, that $TR(\mu_{T/S}) \circ_S TR(\mu_S)$ is of the form $123 \cdot u_{T/S}$ where $u_{T/S} = 0$ or $u_{T/S} \in \chi(V_{T/S} \setminus \{S\})$. The image of this element under π_k is 0 for every $k \geq 4$. If $k = 3$, then we have $123 \cdot \pi_3(123 \cdot u_{T/S}) = 1233 \cdot \pi_3(u_{T/S}) = 0$.

— Suppose now that $|T/S| = 2$ and $|S| \neq 2$. Because $b_T = 1$, we have $r(S) = 1$. We then obtain that $TR(\mu_{T/S}) = S\alpha$ for some $1 \leq \alpha \leq |T|$. Therefore, by induction hypothesis, the composite $TR(\mu_{T/S}) \circ_S TR(\mu_S)$ is given by $1\Sigma \cdot u_S \cdot \alpha$, where Σ is the second vertex of S. If $\alpha \neq 2$, then $\Sigma = 2$ and u_S contains an occurrence of 2. Taking the image of such element under π_k for every $k \geq 3$ gives 0. If $\alpha = 2$, then $\Sigma = 3$. The image under π_k of the resulting composite gives 0 for every $k \geq 4$. If $k = 3$, then the resulting term in the sum is $123 \cdot \pi_3(13 \cdot u_S \cdot 2) = 1233 \cdot \pi_3(u_S) = 0$.

The lemma is proved. □

This lemma allows us to compute ∂^0.

Lemma 2.3.6. *We have the following identity:*

$$\partial^0(\Sigma^{-1}\underline{0}) = \begin{array}{c} \Sigma^{-1}\underline{0} \\ | \\ \Sigma^{-1}\underline{0} \end{array}.$$

Proof. For every canonical n-tree T with $n \geq 3$, we have that $\Lambda\mu_T(\Sigma^{-1}\underline{0}) = 0$. Indeed, by Lemma 2.3.5, in this case, the number 2 occurs at least two times in the surjection $TR(\mu_T)$, so that its coevaluation on $\underline{0}$ gives 0 by definition of the interval cuts operations. There only remains the case $n = 2$. The associated canonical tree is $T = \begin{array}{c} \textcircled{2} \\ | \\ \textcircled{1} \end{array}$, which gives $TR(\mu_T) = (12)$ by definition. By definition of the interval cuts operations, the coevaluation of the surjection (12) on $\underline{0}$ is $\underline{0} \otimes \underline{0}$, which gives the result. $\square$

For every $n, k \geq 0$, let $\partial^n_{(k)}$ be the composite of ∂^n with the projection on trees with $k+1$ vertices. Our goal is to compute $\partial^n_{(k)}$ by induction on $n, k \geq 0$. Recall, from after Proposition 2.1.32, the two morphisms $\varphi^0_n : N_*(\Delta^n) \longrightarrow N_*(\Delta^n)$ and $h^0_n : N_*(\Delta^n) \longrightarrow N_{*+1}(\Delta^n)$, which satisfy

$$dh^0_n + h^0_n d = id_{N_*(\Delta^n)} - \varphi^0_n.$$

We keep the notations φ^0_n and h^0_n for the two induced morphisms on $\Sigma^{-1}N_*(\Delta^n)$, which also satisfy the same homotopy relation. We extend such morphisms on the tensor algebra of $\Sigma^{-1}N_*(\Delta^n)$ by

$$(\varphi^0_n)(x_1 \otimes \cdots \otimes x_p) = \varphi^0_n(x_1) \otimes \cdots \otimes \varphi^0_n(x_p);$$

$$(h^0_n)(x_1 \otimes \cdots \otimes x_p) = \sum_{i=1}^{p} \pm \varphi^0_n(x_1) \otimes \cdots \otimes \varphi^0_n(x_{i-1}) \otimes h^0_n(x_i) \otimes x_{i+1} \otimes \cdots \otimes x_p,$$

for every $x_1, \ldots, x_p \in \Sigma^{-1}N_*(\Delta^n)$. We extend φ^0_n and h^0_n on $\mathcal{B}race(\Sigma^{-1}N_*(\Delta^n))$ by setting, for every canonical tree T and $X \in (\Sigma^{-1}N_*(\Delta^n))^{\otimes |T|}$,

$$\Phi^0_n(T \otimes X) = T \otimes \varphi^0_n(X);$$

$$H^0_n(T \otimes X) = T \otimes h^0_n(X).$$

These definitions are extended to any tree T by symmetry. We then obtain the homotopy relation:

$$\partial^n_{(0)} H^0_n + H^0_n \partial^n_{(0)} = id - \Phi^0_n.$$

Theorem 2.3.7. *Let $n, k \geq 1$. Then*

$$\partial^n_{(k)}(\Sigma^{-1}\underline{0 \cdots n}) = -H^0_n \left(\sum_{\substack{p+q=k \\ p \neq 0}} \partial^n_{(p)} \partial^n_{(q)} (\Sigma^{-1}\underline{0 \cdots n}) \right).$$

In particular, we have the induction relation:

$$\partial^n_{(k)}(\Sigma^{-1}\underline{0 \cdots n}) = H^0_n d^0 \partial^{n-1}_{(k)}(\Sigma^{-1}\underline{0 \cdots (n-1)}) - H^0_n \left(\sum_{\substack{p+q=k \\ p,q \neq 0}} \partial^n_{(p)} \partial^n_{(q)} (\Sigma^{-1}\underline{0 \cdots n}) \right).$$

Starting from Lemma 2.3.6, this theorem allows us to compute the elements $\partial^n_{(k)}(\Sigma^{-1}\underline{0 \cdots n})$ by induction on $n, k \geq 0$.

Proof. We have

$$\partial^n_{(0)}\partial^n_{(k)}(\Sigma^{-1}\underline{0\cdots n}) = -\sum_{\substack{p+q=k\\ p\neq 0}} \partial^n_{(p)}\partial^n_{(q)}(\Sigma^{-1}\underline{0\cdots n}).$$

Applying H^0_n on this equality gives

$$\begin{aligned}\partial^n_{(k)}(\Sigma^{-1}\underline{0\cdots n}) = -H^0_n &\left(\sum_{\substack{p+q=k\\ p\neq 0}} \partial^n_{(p)}\partial^n_{(q)}(\Sigma^{-1}\underline{0\cdots n})\right)\\ &+\Phi^0_n\left(\sum_{\substack{p+q=k\\ p\neq 0}} \partial^n_{(p)}\partial^n_{(q)}(\Sigma^{-1}\underline{0\cdots n})\right) - \partial^n_{(0)}H^0_n\partial^n_{(k)}(\Sigma^{-1}\underline{0\cdots n})\end{aligned}$$

The second term on the right hand-side vanishes, since the differentials ∂^n are compatible with the simplicial structure of $\widehat{\mathcal{B}race}(\Sigma^{-1}N_*(\Delta^\bullet))$ defined tensor-wise and $\Phi^0_n(\Sigma^{-1}\underline{0\cdots n}) = 0$ since $n \geq 1$. We now deal with the last term. Let T be a canonical tree with $|T| = k+1$ and $b_T = 1$. By Lemma 2.3.5, there exists $u_T \in \chi([\![2, k+1]\!])$ such that

$$TR(\mu_T) = 12 \cdot u_T.$$

Then, up to a sign, the elements occurring in each terms of $TR(\mu_T)(\underline{0\cdots n})$ are on the form

$$\underline{0\cdots k} \otimes \underline{x} \otimes X,$$

where $\underline{x} \in N_*(\Delta^n)$ has a length which is at least 2, and X is a tensor product of elements in $N_*(\Delta^n)$. The image of such elements by h^0_n is 0. We thus deduce that $H^0_n\partial^n_{(k)}(\Sigma^{-1}\underline{0\cdots n}) = 0$, which proves the first formula.

We now prove the induction relation. For every $n, k \geq 1$, we have

$$\partial^n_{(k)}(\Sigma^{-1}\underline{0\cdots n}) = -H^0_n\partial^n_{(k)}\partial^n_{(0)}(\Sigma^{-1}\underline{0\cdots n}) - H^0_n\left(\sum_{\substack{p+q=k\\ p,q\neq 0}} \partial^n_{(p)}\partial^n_{(q)}(\Sigma^{-1}\underline{0\cdots n})\right).$$

We only need to compute the first term. We have

$$\partial^n_{(k)}\partial^n_{(0)}(\Sigma^{-1}\underline{0\cdots n}) = -\sum_{i=0}^{n}(-1)^i d^i\partial^{n-1}_{(k)}(\Sigma^{-1}\underline{0\cdots(n-1)}).$$

For every $1 \leq i \leq n$ we have $H^0_n d^i = d^i H^0_{n-1}$. From the first formula that we have proved, we deduce that the element $\partial^{n-1}_{(k)}(\Sigma^{-1}\underline{0\cdots(n-1)})$ is in the image of H^0_{n-1}. Since $H^0_{n-1}H^0_{n-1} = 0$, we obtain at the end

$$H^0_n\partial^n_{(k)}\partial^n_{(0)}(\Sigma^{-1}\underline{0\cdots n}) = H^0_n d^0\partial^{n-1}_{(k)}(\Sigma^{-1}\underline{0\cdots(n-1)}),$$

which proves the theorem. $\square$

Corollary 2.3.8. *Let $n \geq 0$. Then*

$$\partial^n_{(1)}(\Sigma^{-1}\underline{0\cdots n}) = \sum_{k=0}^{n}(-1)^k \begin{array}{c} \Sigma^{-1}\underline{k\cdots n} \\ | \\ \Sigma^{-1}\underline{0\cdots k} \end{array}.$$

Proof. We prove the corollary by induction on $n \geq 0$. Lemma 2.3.6 proves the case $n = 0$. We now suppose that $n \geq 1$. Theorem 2.3.7 gives

$$\partial^n_{(1)}(\Sigma^{-1}\underline{0\cdots n}) = H^0_n d^0 \partial^{n-1}_{(1)}(\Sigma^{-1}\underline{0\cdots(n-1)}).$$

By induction hypothesis on n, we have

$$d^0\partial^{n-1}_{(1)}(\Sigma^{-1}\underline{0\cdots(n-1)}) = -\sum_{k=1}^{n}(-1)^k \begin{array}{c} \Sigma^{-1}\underline{k\cdots n} \\ | \\ \Sigma^{-1}\underline{1\cdots k} \end{array}.$$

This gives

$$\partial^n_{(1)}(\Sigma^{-1}\underline{0\cdots n}) = \sum_{k=1}^{n}(-1)^k \begin{array}{c} \Sigma^{-1}\underline{k\cdots n} \\ | \\ \Sigma^{-1}\underline{0\cdots k} \end{array} + \begin{array}{c} \Sigma^{-1}\underline{0\cdots n} \\ | \\ \Sigma^{-1}\underline{0} \end{array},$$

which gives the result. $\square$

We can also compute the differentials ∂^1, ∂^2 and ∂^3. In the following corollary, we set

$$\Sigma^{-1}\underline{x}\overline{\circledcirc}\Sigma^{-1}\underline{y} = \Sigma^{-1}\underline{x} + \Sigma^{-1}\underline{y} + \sum_{k\geq 1} \begin{array}{ccc} \Sigma^{-1}\underline{y} & \overset{k}{\cdots} & \Sigma^{-1}\underline{y} \\ \diagdown & & \diagup \\ & \Sigma^{-1}\underline{x} & \end{array},$$

for every degree 1 elements $\underline{x}, \underline{y} \in N_*(\Delta^n)$. Note that the operation $\overline{\circledcirc}$ corresponds to the circular product $\circledcirc$ defined in [Ver23, Remark 2.20] in the brace algebra $\widehat{\mathcal{B}race}(\Sigma^{-1}N_*(\Delta^n))$. In particular, the product $\overline{\circledcirc}$ is associative. This operation is reviewed in details in the beginning of §2.4.2. In order to write shorter formulas, we also put a weight on the arrows of our trees. We precisely set, for every $\underline{x} \in N_*(\Delta^n)$, for every tree $T_1, \ldots, T_k \in \widehat{\mathcal{B}race}(\Sigma^{-1}N_*(\Delta^n))$ and for every integers $r_1, \ldots, r_k \geq 1$,

$$\begin{array}{ccc} T_1 & \cdots & T_k \\ \scriptstyle{(r_1)}\diagdown & & \diagup\scriptstyle{(r_k)} \\ & \Sigma^{-1}\underline{x} & \end{array} = \begin{array}{ccccccc} T_1 & \overset{r_1}{\cdots} & T_1 & \cdots & T_k & \overset{r_k}{\cdots} & T_k \\ & & \diagdown & & \diagup & & \\ & & & \Sigma^{-1}\underline{x} & & & \end{array}.$$

If $r_i = 1$ for some i, we remove the weight from the arrow.

In the following corollary, we drop the desuspension Σ^{-1} on basis elements of $\Sigma^{-1}N_*(\Delta^n)$.

Corollary 2.3.9. *We have the following formulas in* $\widehat{\mathcal{B}race}(\Sigma^{-1}N_*(\Delta^n))$*:*

- $$\partial^1(\underline{01}) = \underline{0} - \underline{1} - \begin{array}{c}\underline{1}\\ | \\ \underline{01}\end{array} + \sum_{k\geq 1} \begin{array}{c}\underline{01}\\ \textcircled{k} \\ \underline{0}\end{array}\ ;$$

- $$\partial^2(\underline{012}) = \underline{02} - \underline{01} - \underline{12} + \begin{array}{c}\underline{2}\\ | \\ \underline{012}\end{array} - \sum_{k\geq 1} \begin{array}{c}\underline{12}\\ \textcircled{k} \\ \underline{01}\end{array} + \sum_{i,j\geq 0} \begin{array}{ccccc}\underline{02} & & \underline{012} & & \underline{01}\overline{\circledcirc}\underline{12}\\ & \textcircled{i} & | & \textcircled{j} & \\ & & \underline{0} & & \end{array}\ ;$$

- $$\begin{aligned}\partial^3(\underline{0123}) = {} & \underline{023} - \underline{123} + \underline{013} - \underline{012} - \begin{array}{c}\underline{3}\\ | \\ \underline{0123}\end{array}\\ & + \sum_{k\geq 1} \begin{array}{c}\underline{23}\\ \textcircled{k} \\ \underline{012}\end{array} - \sum_{i,j\geq 0} \begin{array}{ccccc}\underline{13} & & \underline{123} & & \underline{12}\overline{\circledcirc}\underline{23}\\ & \textcircled{i} & | & \textcircled{j} & \\ & & \underline{01} & & \end{array}\\ & - \sum_{i,j,k\geq 0} \begin{array}{ccccc}\underline{03} & \underline{023} & \underline{02}\overline{\circledcirc}\underline{23} & \underline{012}\circledcirc(1+\underline{23}) & \underline{01}\overline{\circledcirc}\underline{12}\overline{\circledcirc}\underline{23}\\ \textcircled{i} & \backslash & \textcircled{j} & / & \textcircled{k}\\ & & \underline{0} & & \end{array}\\ & + \sum_{i,j,k,l,m\geq 0} \begin{array}{ccccc}\underline{03} & \underline{013} & \underline{01}\overline{\circledcirc}\underline{13} & \begin{array}{ccccc}\underline{13} & & \underline{123} & & \underline{12}\overline{\circledcirc}\underline{23}\\ & \textcircled{l} & | & \textcircled{m} & \\ & & \underline{01} & & \end{array} & \underline{01}\overline{\circledcirc}\underline{12}\overline{\circledcirc}\underline{23}\\ \textcircled{i} & \backslash & \textcircled{j} & / & \textcircled{k}\\ & & \underline{0} & & \end{array}\\ & + \sum_{i,j,k\geq 0} \begin{array}{ccccc}\underline{03} & \underline{013} & \underline{01}\overline{\circledcirc}\underline{13} & \underline{123} & \underline{01}\overline{\circledcirc}\underline{12}\overline{\circledcirc}\underline{23}\\ \textcircled{i} & \backslash & \textcircled{j} & / & \textcircled{k}\\ & & \underline{0} & & \end{array}\\ & + \sum_{i,j\geq 0} \begin{array}{ccccc}\underline{03} & & \underline{0123} & & \underline{01}\overline{\circledcirc}\underline{12}\overline{\circledcirc}\underline{23}\\ & \textcircled{i} & | & \textcircled{j} & \\ & & \underline{0} & & \end{array}\ .\end{aligned}$$

Proof. We first compute ∂^1. Lemma 2.3.6 and Corollary 2.3.8 give

$$\partial^1(\underline{0}) = \begin{array}{c}\underline{0}\\ | \\ \underline{0}\end{array}\ ;\ \partial^1(\underline{1}) = \begin{array}{c}\underline{1}\\ | \\ \underline{1}\end{array}\ ;\ \partial^1_{(1)}(\underline{01}) = \begin{array}{c}\underline{01}\\ | \\ \underline{0}\end{array} - \begin{array}{c}\underline{1}\\ | \\ \underline{01}\end{array}\ .$$

We compute $\partial^1_{(k)}(\underline{01})$ by induction on $k \geq 1$. We give the details for the case $k = 2$.

By Theorem 2.3.7, we have

$$\partial^1_{(2)}(\underline{01}) = -H^0_1\partial^1_{(2)}\partial^1_{(0)}(\underline{01}) - H^0_1\partial^1_{(1)}\partial^1_{(1)}(\underline{01}).$$

Because the differentials ∂^n's preserve the face maps, we have

$$\begin{array}{rcl} \partial^1_{(2)}\partial^1_{(0)}(\underline{01}) & = & d^0\partial^0_{(2)}(\underline{0}) - d^1\partial^0_{(2)}(\underline{0}) \\ & = & 0, \end{array}$$

since $\partial^0_{(2)} = 0$. Now, by the Leibniz rule in $\mathcal{B}race(\Sigma^{-1}N_*(\Delta^1))$,

$$\partial^1_{(1)}\partial^1_{(1)}(\underline{01}) = \begin{array}{c} \underline{01} \\ | \\ \partial^1_{(1)}(\underline{0}) \end{array} - \begin{array}{c} \partial^1_{(2)}(\underline{01}) \\ | \\ \underline{0} \end{array} - \begin{array}{c} \underline{1} \\ | \\ \partial^1_{(2)}(\underline{01}) \end{array} - \begin{array}{c} \partial^1_{(2)}(\underline{1}) \\ | \\ \underline{01} \end{array}.$$

These terms give

$$\begin{array}{c} \underline{01} \\ | \\ \partial^1_{(1)}(\underline{0}) \end{array} = \begin{array}{c} \underline{01} \\ | \\ \underline{0} \\ | \\ \underline{0} \end{array} + \begin{array}{ccc} \underline{0} & & \underline{01} \\ & \diagdown\ \diagup & \\ & \underline{0} & \end{array} + \begin{array}{ccc} \underline{01} & & \underline{0} \\ & \diagdown\ \diagup & \\ & \underline{0} & \end{array};$$

$$- \begin{array}{c} \partial^1_{(2)}(\underline{01}) \\ | \\ \underline{0} \end{array} = - \begin{array}{c} \underline{01} \\ | \\ \underline{0} \\ | \\ \underline{0} \end{array} + \begin{array}{c} \underline{1} \\ | \\ \underline{01} \\ | \\ \underline{0} \end{array};$$

$$\begin{array}{rl} - \begin{array}{c} \underline{1} \\ | \\ \partial^1_{(2)}(\underline{01}) \end{array} = & - \begin{array}{c} \underline{1} \\ | \\ \underline{01} \\ | \\ \underline{0} \end{array} + \begin{array}{c} \underline{1} \\ | \\ \underline{1} \\ | \\ \underline{01} \end{array} \boxed{- \begin{array}{ccc} \underline{1} & & \underline{01} \\ & \diagdown\ \diagup & \\ & \underline{0} & \end{array}} - \begin{array}{ccc} \underline{01} & & \underline{1} \\ & \diagdown\ \diagup & \\ & \underline{0} & \end{array} \\ & + \begin{array}{ccc} \underline{1} & & \underline{1} \\ & \diagdown\ \diagup & \\ & \underline{01} & \end{array} - \begin{array}{ccc} \underline{1} & & \underline{1} \\ & \diagdown\ \diagup & \\ & \underline{01} & \end{array}; \end{array}$$

$$- \begin{array}{c} \partial^1_{(2)}(\underline{1}) \\ | \\ \underline{01} \end{array} = - \begin{array}{c} \underline{1} \\ | \\ \underline{1} \\ | \\ \underline{01} \end{array}.$$

The boxed tree is the only tree which gives a non-zero element when applying H^0_1. This gives

$$\partial^1_{(2)}(\underline{01}) = \begin{array}{c} \underline{01} \\ \textcircled{2} \\ \underline{0} \end{array}.$$

We now suppose that $k \geq 3$. By definition, and since $\partial^1_{(k-1)}\partial^1_{(0)}(\underline{01}) = 0$ because $\partial^0_{(k-1)} = 0$, we have

$$\partial^1_{(k)}(\underline{01}) = -H^0_1\partial^1_{(k-1)}\partial^1_{(1)}(\underline{01}) - H^0_1\partial^1_{(1)}\partial^1_{(k-1)}(\underline{01}) - \sum_{\substack{p+q=k \\ p,q\neq 0,1}} H^0_1\partial^1_{(p)}\partial^1_{(q)}(\underline{01}).$$

By induction hypothesis, for every $p, q \neq 0, 1$ such that $p + q = k$, the term $\partial^1_{(p)}\partial^1_{(q)}(\underline{01})$ will only give trees with as vertices $\underline{0}$ or $\underline{01}$, so that $H^0_1\partial^1_{(p)}\partial^1_{(q)}(\underline{01}) = 0$. We now look at the remaining terms. Since $k - 1 \geq 2$, we have by induction hypothesis and by the Leibniz rule

$$\partial^1_{(1)}\partial^1_{(k-1)}(\underline{01}) = \sum_{p+q+r=k-1} \begin{array}{ccccc} & & \underline{01} & & \\ & & \textcircled{q} & & \\ \underline{01} & & \underline{0} & & \underline{01} \\ & \textcircled{p} & | & \textcircled{r} & \\ & & \underline{0} & & \end{array}$$

$$+ \sum_{p+q=k-1} \begin{array}{ccccc} & & \underline{01} & & \\ & & \textcircled{q} & & \\ \underline{01} & & \underline{0} & & \underline{01} \\ & \textcircled{p} & | & \textcircled{r} & \\ & & \underline{0} & & \end{array} - \sum_{p+q=k-1} \begin{array}{ccccc} & & \underline{1} & & \\ & & | & & \\ \underline{01} & & \underline{01} & & \underline{01} \\ & \textcircled{p} & | & \textcircled{q} & \\ & & \underline{0} & & \end{array},$$

which gives $H^0_1\partial^1_{(1)}\partial^1_{(k-1)}(\underline{01}) = 0$. We finally have

$$\partial^1_{(k-1)}\partial^1_{(1)}(\underline{01}) = -\begin{array}{c} \underline{01} \\ \textcircled{k-1} \\ \underline{0} \\ | \\ \underline{0} \end{array} - \sum_{p+q=k-2} \begin{array}{ccccc} & & \underline{1} & & \\ & & | & & \\ \underline{01} & & \underline{01} & & \underline{01} \\ & \textcircled{p} & | & \textcircled{q} & \\ & & \underline{0} & & \end{array} - \sum_{p+q=k-1} \begin{array}{ccccc} \underline{01} & & \underline{1} & & \underline{01} \\ & \textcircled{p} & | & \textcircled{q} & \\ & & \underline{0} & & \end{array}.$$

All the terms occurring in the right hand-side give elements in the kernel of H^0_1, except for the last sum with $p = 0$ and $q = k - 1$, which gives

$$\partial^1_{(k)}(\underline{01}) = \begin{array}{c} \underline{01} \\ \textcircled{k} \\ \underline{0} \end{array}.$$

The computation of ∂^1 is proved. We now compute ∂^2. By Lemma 2.3.6, we have

$$\partial^2(\underline{0}) = \begin{array}{c} \underline{0} \\ | \\ \underline{0} \end{array} \; ; \; \partial^2(\underline{1}) = \begin{array}{c} \underline{1} \\ | \\ \underline{1} \end{array} \; ; \; \partial^2(\underline{2}) = \begin{array}{c} \underline{2} \\ | \\ \underline{2} \end{array}.$$

By Corollary 2.3.8, we have

$$\partial^2_{(1)}(\underline{01}) = \begin{array}{c}\underline{01}\\ | \\ \underline{0}\end{array} - \begin{array}{c}\underline{1}\\ | \\ \underline{01}\end{array} \ ;\ \partial^2_{(1)}(\underline{02}) = \begin{array}{c}\underline{02}\\ | \\ \underline{0}\end{array} - \begin{array}{c}\underline{2}\\ | \\ \underline{02}\end{array} \ ;\ \partial^2_{(1)}(\underline{12}) = \begin{array}{c}\underline{12}\\ | \\ \underline{1}\end{array} - \begin{array}{c}\underline{2}\\ | \\ \underline{12}\end{array},$$

and

$$\partial^2_{(1)}(\underline{012}) = \begin{array}{c}\underline{012}\\ | \\ \underline{0}\end{array} - \begin{array}{c}\underline{12}\\ | \\ \underline{01}\end{array} + \begin{array}{c}\underline{2}\\ | \\ \underline{012}\end{array}.$$

As before, we can compute by hand $\partial^2_{(2)}(\underline{012})$ by using Theorem 2.3.7. We obtain

$$\partial^1_{(2)}(\underline{012}) = -\begin{array}{ccccc}\underline{12}&&&&\underline{12}\\ &\diagdown&&\diagup&\\ &&\underline{01}&&\end{array} + \begin{array}{ccccc}\underline{012}&&&&(\underline{01}+\underline{12})\\ &\diagdown&&\diagup&\\ &&\underline{0}&&\end{array} + \begin{array}{ccccc}\underline{02}&&&&\underline{012}\\ &\diagdown&&\diagup&\\ &&\underline{0}&&\end{array}.$$

We now compute $\partial^2_{(k)}(\underline{012})$ for every $k \geq 3$. We use that

$$\partial^2_{(k)}(\underline{012}) = -H^0_2\partial^2_{(k)}\partial^2_{(0)}(\underline{012}) - H^0_2\partial^2_{(k-1)}\partial^2_{(1)}(\underline{012}) - \sum_{\substack{p+q=k\\ q\neq 0,1\\ p\neq 0}} H^0_2\partial^2_{(p)}\partial^2_{(q)}(\underline{012}).$$

We have

$$-H^0_2\partial^2_{(k)}\partial^2_{(0)}(\underline{012}) = -\begin{array}{c}\underline{12}\\ k \\ \underline{01}\end{array} + \begin{array}{ccccc}\underline{012}&&&&\underline{12}\\ &\diagdown&&k-1&\\ &&\underline{0}&&\end{array}$$

by induction hypothesis. We now compute $H^0_2\partial^2_{(k-1)}\partial^2_{(1)}(\underline{012})$. We have

$$\begin{array}{rcl} -H^0_2\partial^2_{(k-1)}\left(\begin{array}{c}\underline{012}\\ | \\ \underline{0}\end{array}\right) &=& 0,\\ H^0_2\partial^2_{(k-1)}\left(\begin{array}{c}\underline{12}\\ | \\ \underline{01}\end{array}\right) &=& \begin{array}{ccccc}\underline{012}&&&&\underline{01}\\ &\diagdown&&k-1&\\ &&\underline{0}&&\end{array}, \end{array}$$

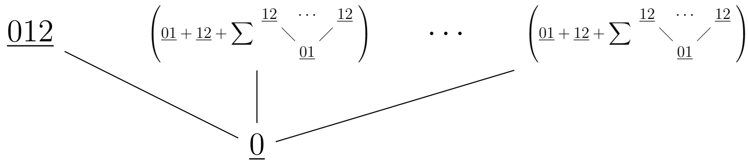
and

$$-H^0_2\partial^2_{(k-1)}\left(\begin{array}{c}\underline{2}\\ | \\ \underline{012}\end{array}\right) =$$

$$\sum \begin{array}{ccccccc}\underline{02} & \cdots & \underline{02} & \underline{012} & \left(\underline{01}+\underline{12}+\sum \begin{array}{ccc}\underline{12}&\cdots&\underline{12}\\ \diagdown&&\diagup\\ &\underline{01}&\end{array}\right) & \cdots & \left(\underline{01}+\underline{12}+\sum \begin{array}{ccc}\underline{12}&\cdots&\underline{12}\\ \diagdown&&\diagup\\ &\underline{01}&\end{array}\right)\\ &&& \underline{0} &&& \end{array}$$

where we sum over all such trees with $k+1$ vertices which contain at least one element $\underline{02}$. Finally, for every p, q such that $p+q=k, q \neq 0, 1$ and $p \neq 0$, we have

$$-H_2^0 \partial^2_{(p)} \partial^2_{(q)}(\underline{012}) = \sum$$



where we sum over all such trees with p elements $\underline{01}$ and $q-1$ elements $\underline{12}$. This concludes the proof of the computation of ∂^2.

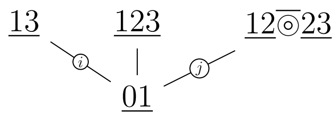
We only give the ideas for the computation of ∂^3. By induction on $k \geq 0$, we first two lines are given by $\partial^3_{(k)}\partial^3_{(0)}(\underline{0123})$. The tree

$$\begin{array}{c}\underline{3}\\ | \\ \underline{0123}\end{array}$$

will give trees of $\partial^3_{(k-1)}(\underline{0123})$ with $\underline{0}$ as root on which we add a branch linked to the root with $\underline{03}$ as non-root vertex. We thus can focus to trees with no elements $\underline{03}$. The trees with no element $\underline{03}$ of the third line are obtained from the differentiation of the trees of the form

$$\begin{array}{c}\underline{23}\\ \textcircled{\scriptsize i} \\ \underline{012}\end{array}$$

for some $1 \leq i \leq k-1$. The trees of the three last lines with no element $\underline{03}$ is obtained by the differentiation of trees of the form



for some $i, j \geq 0$ such that $1 \leq i+j \leq k-2$.

The lemma is proved. $\square$

2.3.3 A morphism from $\mathcal{P}re\mathcal{L}ie_\infty$ to $\mathcal{B}race \underset{\mathrm{H}}{\otimes} B^c(\Lambda^{-1}\mathcal{B}race^\vee)$

In this subsection, we construct a morphism $\mathcal{P}re\mathcal{L}ie_\infty \longrightarrow \mathcal{B}race \underset{\mathrm{H}}{\otimes} B^c(\Lambda^{-1}\mathcal{B}race^\vee)$ which will give, together with Theorem 2.3.3, a morphism of operads $\mathcal{P}re\mathcal{L}ie_\infty \longrightarrow \mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$.

Recall that, for every operad $\mathcal{P}$ such that $\mathcal{P}(n)$ is finite dimensional for every $n \geq 0$, we have a morphism of operads

$$\begin{array}{ccc}\mathcal{L}ie_\infty & \longrightarrow & \mathcal{P} \otimes B^c(\Lambda^{-1}\mathcal{P}^\vee) \\ \Sigma^{n-1} & \longmapsto & \sum_{x \in \mathcal{B}(n)} x \otimes \Sigma^{n-1} x^\vee\end{array}$$

where $\mathcal{B}(n)$ denotes a basis of $\mathcal{P}(n)$, and where $\mathcal{L}ie_\infty = B^c(\Lambda^{-1}\mathcal{C}om^\vee)$ is the operad which governs Lie algebras up to homotopy. Recall also that we have a morphism of operads $\mathcal{L}ie_\infty \longrightarrow \mathcal{P}re\mathcal{L}ie_\infty$ given by the morphism of symmetric sequences

$$\begin{array}{ccc} \mathrm{Perm} & \longrightarrow & \mathcal{C}om \\ e_1^n & \longmapsto & 1 \end{array}.$$

Lemma 2.3.10. *There exists an explicit lift of the diagram*

$$\begin{tikzcd} \mathcal{L}ie_\infty \arrow[r] \arrow[d] & \mathcal{B}race \underset{\mathrm{H}}{\otimes} B^c(\Lambda^{-1}\mathcal{B}race^\vee) \\ \mathcal{P}re\mathcal{L}ie_\infty \arrow[ur, dashed, "\exists"'] & \end{tikzcd}.$$

Proof. Giving a morphism $\mathcal{P}re\mathcal{L}ie_\infty \longrightarrow \mathcal{B}race \underset{\mathrm{H}}{\otimes} B^c(\Lambda^{-1}\mathcal{B}race^\vee)$ is equivalent to giving a Maurer-Cartan element f in the pre-Lie algebra $\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathrm{Perm}}^\vee, \mathcal{B}race \underset{\mathrm{H}}{\otimes} B^c(\mathcal{B}race^\vee))$ (see for instance [LV12, Theorem 6.5.10]). By symmetry, it is sufficient to give the image of e_1^n for every $n \geq 1$. We set

$$f(e_1^n) = -\sum_{\substack{T \in \mathcal{PRT}(n) \\ r(T)=1}} T \otimes \Sigma^{-1}T^\vee.$$

We check that $d(f)(e_1^n) + (f \star f)(e_1^n) = 0$, where $\star$ denotes the pre-Lie product of $\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathrm{Perm}}^\vee, \mathcal{B}race \underset{\mathrm{H}}{\otimes} B^c(\mathcal{B}race^\vee))$. We first have

$$d(f)(e_1^n) = d(f(e_1^n)) = -\sum_{\substack{T \in \mathcal{PRT}(n) \\ r(T)=1}} \sum_{S \subset T} T \otimes (\Sigma^{-1}(T/S)^\vee \circ_S \Sigma^{-1}S^\vee).$$

We now compute $(f \star f)(e_1^n)$. Recall that

$$\Delta_1(e_1^n) = \sum_{\substack{p+q=n+1 \\ p,q \geq 2}} e_1^p \otimes e_1^q;$$

$$\forall k \neq 1, \Delta_k(e_1^n) = \sum_{\substack{p+q=n+1 \\ p,q \geq 2}} \sum_{i=1}^{q} e_1^p \otimes e_i^q.$$

The Δ_1 part gives

$$\sum_{\substack{p+q=n+1 \\ p,q \geq 2}} \sum_{\omega \in Sh_*(q,1,\ldots,1)} \sum_{\substack{U \in \mathcal{PRT}(p) \\ r(U)=1}} \sum_{\substack{V \in \mathcal{PRT}(q) \\ r(V)=1}} \omega \cdot (U \circ_1 V) \otimes \omega \cdot (\Sigma^{-1}U^\vee \circ_1 \Sigma^{-1}V^\vee),$$

and the Δ_k's part for $k \neq 1$ gives

$$\sum_{\substack{p+q=n+1\\ p,q\geq 2}} \sum_{k=2}^{p} \sum_{\omega \in Sh_*(1,\ldots,\underset{k}{q},\ldots,1)} \sum_{\substack{U \in \mathcal{PRT}(p)\\ r(U)=1}} \sum_{V \in \mathcal{PRT}(q)} \omega \cdot (U \circ_k V) \otimes \omega \cdot (\Sigma^{-1} U^\vee \circ_k \Sigma^{-1} V^\vee).$$

Let $T \in \mathcal{PRT}(n)$ be such that $r(T) = 1$. Consider a term $\omega \cdot (U \circ_k V)$ occurring in one of the two previous sums, with $k \geq 1$, $\omega \in Sh_*(1,\ldots,\underset{k}{q},\ldots,1), U \in \mathcal{PRT}(p), V \in \mathcal{PRT}(q)$, such that T occurs in the expansion of $\omega \cdot (U \circ_k V)$. We see U and V as $U \in \mathcal{PRT}(1 < \cdots < k-1 < V < k+q < \cdots < p)$ and $V \in \mathcal{PRT}(k < \cdots < k+q-1)$. Because $\omega \in Sh_*(1,\ldots,\underset{k}{q},\ldots,1)$, the composite $\omega \cdot (U \circ_k V)$ is equal to the composite $(\omega \cdot U) \circ_V (\omega \cdot V)$ where $\omega \cdot U$ is U seen in $\mathcal{PRT}(1 < \cdots < k-1 < V < \omega(k+q) < \cdots < \omega(p))$ and $\omega \cdot V$ is V seen in $\mathcal{PRT}(k = \omega(k) < \cdots < \omega(k+q-1))$. Thus, by definition of the operadic composition in $\mathcal{B}race$, the tree $\omega \cdot V$ can be seen as a subtree $S \subset T$ such that $T/S = \omega \cdot U$.

In the converse direction, let $S \subset T$. Let $k = min(V_S)$ and $q = |S|$. Let $\omega_S : [\![k, k+q-1]\!] \longrightarrow V_S$ and $\omega_{T/S} : [\![1,n]\!] \setminus [\![k+1, k+q-1]\!] \longrightarrow V_{T/S}$ be the unique order preserving maps between the two considered finite sets. Then, by definition, $\omega = \omega_{T/S} \circ_k \omega_S \in Sh_*(1,\ldots,\underset{k}{q},\ldots,1)$. We finally set $U = \omega^{-1} \cdot (T/S)$ and $V = \omega^{-1} \cdot S$. Because $T/S \circ_S S$ obviously contains the tree T, we have that T occurs in the composite $\omega \cdot (U \circ_k V)$.

We thus have proved that

$$(f \star f)(e_1^n) = \sum_{\substack{T \in \mathcal{PRT}(n)\\ r(T)=1}} T \otimes \left(\sum_{S \subset T} \Sigma^{-1}(T/S)^\vee \circ_S \Sigma^{-1} S^\vee \right).$$

The identity $d(f)(e_1^n) + (f \star f)(e_1^n) = 0$ follows. □

We now prove Theorem D.

Theorem 2.3.11. *There exists an operad morphism $\mathcal{P}re\mathcal{L}ie_\infty \longrightarrow \mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$ which fits in a commutative square*

$$\begin{tikzcd} \mathcal{P}re\mathcal{L}ie_\infty \arrow[r] \arrow[d] & \mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E} \arrow[d] \\ \mathcal{P}re\mathcal{L}ie \arrow[r] & \mathcal{B}race \end{tikzcd}.$$

Proof. The morphism $\mathcal{P}re\mathcal{L}ie_\infty \longrightarrow \mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$ is given by the composite of the morphism $\mathcal{P}re\mathcal{L}ie_\infty \longrightarrow \mathcal{B}race \underset{\mathrm{H}}{\otimes} B^c(\Lambda^{-1}\mathcal{B}race^\vee)$ given by Lemma 2.3.10 with the morphism $\mathcal{B}race \underset{\mathrm{H}}{\otimes} B^c(\Lambda^{-1}\mathcal{B}race^\vee) \longrightarrow \mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$ given by applying the morphism $B^c(\Lambda^{-1}\mathcal{B}race^\vee) \longrightarrow \mathcal{E}$ defined in Theorem 2.3.3 on the second tensor. The commutative diagram is an immediate check. □

Corollary 2.3.12. *Every $\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$-algebra L admits the structure of a $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebra.*

Proof. Using that the action of Σ_n on $(\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E})(n)$ is free and the previous theorem, we define, for every $\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$-algebras L, the composite

$$\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, L) \longrightarrow \Gamma(\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}, L) \xleftarrow{\simeq} \mathcal{S}(\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}, L) \longrightarrow L.$$

This gives a $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebra structure on L. □

In particular, if $L = A \otimes \Sigma E$ where A is a brace algebra and E a $\mathcal{E}$-algebra, then $A \otimes \Sigma E$ is a $\Gamma\Lambda\mathcal{PL}_\infty$-algebra. We can compute the weighted braces of $A \otimes \Sigma E$ as follows. Let $l : \mathcal{S}(\mathcal{B}race, A) \longrightarrow A$ be the brace algebra structure on A, and let ∂^E be the twisting morphism on $\mathcal{B}race^c(\Sigma E)$ induced by the $B^c(\Lambda^{-1}\mathcal{B}race^\vee)$-algebra structure of E (see Theorem 2.3.3). Then, for every $a, b_1, \ldots, b_n \in A, x, y_1, \ldots, y_n \in \Sigma E$ and $r_1, \ldots, r_n \geq 0$, we have

$$\begin{aligned} &a \otimes x\{\!\{b_1 \otimes y_1, \ldots, b_n \otimes y_n\}\!\}_{r_1,\ldots,r_n} \\ &= \sum_{\sigma \in Sh(r_1,\ldots,r_n)} \sum_{\substack{T \in \mathcal{PRT}(r+1) \\ T \text{ canonical}}} \pm l(T \otimes a \otimes c_{\sigma(1)} \otimes \cdots \otimes c_{\sigma(r)}) \otimes \partial^E(T^\vee \otimes x \otimes z_{\sigma(1)} \otimes \cdots \otimes z_{\sigma(r)}), \end{aligned}$$

where we have set $r = r_1 + \cdots + r_n$, $c_1, \ldots, c_r = \underbrace{b_1, \ldots, b_1}_{r_1}, \ldots, \underbrace{b_n, \ldots, b_n}_{r_n}$ and $z_1, \ldots, z_r = \underbrace{x_1, \ldots, x_1}_{r_1}, \ldots, \underbrace{x_n, \ldots, x_n}_{r_n}$. The sign is given by the permutation of the c_i's with x and the z_i's, and the permutation of ∂^E with a and the c_i's.

2.4 The simplicial Maurer-Cartan set of a complete brace algebra

The goal of this section is to define the notion of a simplicial Maurer-Cartan set $\mathcal{MC}_\bullet(A)$ associated to a brace algebra A, and to study the homotopy type of this simplicial set. Explicitly, the n-component $\mathcal{MC}_n(A)$ will be defined as the Maurer-Cartan set of $A \otimes \Sigma N^*(\Delta^n)$ for the $\Gamma\widehat{\Lambda\mathcal{PL}}_\infty$-algebra structure given by Corollary 2.3.12.

In §2.4.1, we define the simplicial set $\mathcal{MC}_\bullet(A)$ and prove the first part of Theorem E which asserts that it is a Kan complex.

In §2.4.2, we prove the remaining part of Theorem E, which gives a computation of the connected components and the homotopy groups of $\mathcal{MC}_\bullet(A)$. More precisely, we first compute the connected components, whose computation is similar to [Ver23, Theorem 3.6], before computing the π_1, π_2 and then the π_n for $n \geq 3$.

In §2.4.4, we prove Theorem F, which is a higher version of the Goldman-Millson theorem (see [GM88, §2.4]). Our proof basically follows the proof found in [MR23b,

§6], which will be adapted to our context.

In §2.4.5, we compare our simplicial notion of Maurer-Cartan set defined for complete brace algebras to the notion of simplicial Maurer-Cartan set associated to a complete Lie algebra, and prove that in fact, these two simplicial sets are weakly equivalent.

2.4.1 The simplicial set $\mathcal{MC}_\bullet(A)$

Let A be a complete brace algebra. By Corollary 2.3.12, and using that $N^*(\Delta^n)$ is a $\mathcal{E}$-algebra, we obtain that $A\widehat{\otimes}\Sigma N^*(\Delta^n) = A \otimes \Sigma N^*(\Delta^n)$ is a $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebra with the filtration

$$F_k(A \otimes \Sigma N^*(\Delta^n)) = F_k A \otimes \Sigma N^*(\Delta^n).$$

where we denote by $(F_k A)_{k\geq 1}$ the filtration on A.

Definition 2.4.1. *Let A be a complete brace algebra. Its* simplicial Maurer-Cartan set *is the simplicial set $\mathcal{MC}_\bullet(A)$ such that*

$$\mathcal{MC}_\bullet(A) = \mathcal{MC}(A \otimes \Sigma N^*(\Delta^\bullet)).$$

Proposition 2.4.2. *The previous definition defines a functor $\mathcal{MC}_\bullet$ from the category of complete brace algebras to the category of sets.*

Proof. This follows directly from the fact that every brace algebra morphism $f : A \longrightarrow B$ which preserves the filtrations gives rise to a morphism of $\mathcal{B}race \underset{\mathrm{H}}{\otimes} \Lambda\mathcal{E}$-algebras $f \otimes id : A \otimes \Sigma N^*(\Delta^n) \longrightarrow B \otimes \Sigma N^*(\Delta^n)$ which preserves the filtrations, and then to a strict morphism of $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebras. □

We aim to prove that $\mathcal{MC}_\bullet(A)$ is a Kan complex. We will basically follow the proof of the analogous theorem in [KW21]. Recall from Proposition 2.1.32 that we have morphisms $\varphi_n^i : N^*(\Delta^n) \longrightarrow N^*(\Delta^n)$ and $h_n^i : N^*(\Delta^n) \longrightarrow N^{*-1}(\Delta^n)$ which satisfy

$$dh_n^i + h_n^i d = id - \varphi_n^i.$$

These relations can be carried to $A \otimes \Sigma N^*(\Delta^n)$ by setting $H_n^i = id \otimes \Sigma h_n^i$ and $\Phi_n^i = id \otimes \Sigma\varphi_n^i$. Accordingly, we have

$$dH_n^i + H_n^i d = id - \Phi_n^i.$$

Note that if $x \in \mathcal{MC}(A \otimes \Sigma N^*(\Delta^n))$, then the identity

$$d(x) + \sum_{k\geq 1} x\{\!\{x\}\!\}_k = 0$$

gives, after composing by H_n^i,

$$x = \Phi_n^i(x) + dH_n^i(x) - \sum_{k\geq 1} H_n^i(x\{\!\{x\}\!\}_k).$$

We give the following lemma, which is the analogue of [KW21, Lemma 6.5]. For our needs, we need an analogue valid for every complete $\widehat{\Gamma\Lambda\mathcal{PL}_\infty}$-algebra structure on $A \otimes \Sigma N^*(\Delta^n)$.

Lemma 2.4.3. *Let $n \geq 0$ and $0 \leq i \leq n$. Let $A \in \widehat{\mathrm{dgMod}_\mathbb{K}}$. Consider any complete $\Gamma\Lambda\mathcal{PL}_\infty$-algebra structure on $A \otimes \Sigma N^*(\Delta^n)$ such that $\Phi_n^i : A \otimes \Sigma N^*(\Delta^n) \longrightarrow A \otimes \Sigma N^*(\Delta^n)$ is a strict morphism. Then the map*

$$\begin{array}{ccc} \mathcal{MC}(A \otimes \Sigma N^*(\Delta^n)) & \longrightarrow & (\mathcal{MC}(A \otimes \Sigma N^*(\Delta^n)) \cap Im(\Phi_n^i)) \times Im(dH_n^i) \\ x & \longmapsto & (\Phi_n^i(x), dH_n^i(x)) \end{array}$$

is a bijection.

Proof. We first note that the above map is well defined since Φ_n^i is a strict morphism by hypothesis. Now, let $e \in \mathcal{MC}(A \otimes \Sigma N^*(\Delta^n)) \cap Im(\Phi_n^i)$ and $r \in Im(dH_n^i)$. We set:

$$\begin{cases} \alpha_0 & = & e + r \\ \forall k \geq 0, \alpha_{k+1} & = & e + r - \sum_{l \geq 1} H_n^i(\alpha_k \{\!\{ \alpha_k \}\!\}_l) \end{cases} .$$

This defines a Cauchy sequence $(\alpha_k)_k$. Let α be its limit. We then have

$$\alpha = e + r - \sum_{l \geq 1} H_n^i(\alpha \{\!\{ \alpha \}\!\}_l).$$

From this identity, we deduce $\Phi_n^i \alpha = e$ and $dH_n^i \alpha = r$. We just need to check that $\alpha \in \mathcal{MC}(A \otimes \Sigma N^*(\Delta^n))$. Using the relation

$$dH_n^i + H_n^i d = id - \Phi_n^i,$$

we obtain

$$d(\alpha) = d(e) + \sum_{l \geq 1} H_n^i(d(\alpha \{\!\{ \alpha \}\!\}_l)) - \sum_{l \geq 1} \alpha \{\!\{ \alpha \}\!\}_l + \Phi_n^i \left(\sum_{l \geq 1} \alpha \{\!\{ \alpha \}\!\}_l \right),$$

and then, because Φ_n^i is a strict morphism and that $e \in \mathcal{MC}(A \otimes \Sigma N^*(\Delta^n))$,

$$d(\alpha) + \sum_{l \geq 1} \alpha \{\!\{ \alpha \}\!\}_l = \sum_{l \geq 1} H_n^i(d(\alpha \{\!\{ \alpha \}\!\}_l)).$$

Let $\mathcal{R}(\alpha) = d(\alpha) + \sum_{l \geq 1} \alpha \{\!\{ \alpha \}\!\}_l$. We use the identity

$$\sum_{p+q=l} \alpha \{\!\{ \alpha \}\!\}_p \{\!\{ \alpha \}\!\}_q + \sum_{p+q=l-1} \alpha \{\!\{ \alpha \{\!\{ \alpha \}\!\}_p, \alpha \}\!\}_{1,q} = 0$$

for every $l \geq 1$. We thus have

$$d(\alpha \{\!\{ \alpha \}\!\}_l) = - \sum_{\substack{p+q=l \\ q \neq 0}} \alpha \{\!\{ \alpha \}\!\}_p \{\!\{ \alpha \}\!\}_q - \sum_{p+q=l-1} \alpha \{\!\{ \alpha \{\!\{ \alpha \}\!\}_p, \alpha \}\!\}_{1,q}$$

so that

$$\sum_{l\geq 1} d(\alpha\{\!\{\alpha\}\!\}_l) = -\sum_{q\geq 1}\mathcal{R}(\alpha)\{\!\{\alpha\}\!\}_q - \sum_{q\geq 0}\alpha\{\!\{\mathcal{R}(\alpha),\alpha\}\!\}_q.$$

This leads finally to the identity

$$\mathcal{R}(\alpha) = -\sum_{q\geq 1} H_n^i(\mathcal{R}(\alpha)\{\!\{\alpha\}\!\}_q) - \sum_{q\geq 0} H_n^i(\alpha\{\!\{\mathcal{R}(\alpha),\alpha\}\!\}_{1,q}).$$

It follows from this identity that if $\mathcal{R}(\alpha)\in F_k(A)\otimes\Sigma N^*(\Delta^n)$ for some $k\geq 1$, then $\mathcal{R}(\alpha)\in F_{k+1}(A)\otimes\Sigma N^*(\Delta^n)$. We thus have $\mathcal{R}(\alpha)=0$ so that $\alpha\in\mathcal{MC}(A\otimes\Sigma N^*(\Delta^n))$. We then have a bijection which has as inverse this previous construction. □

Definition 2.4.4. *A simplicial $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebra is a simplicial object in the category $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$. A simplicial $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebra A is* strict *if the face and degeneracy maps of A are strict morphisms of $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebras. A morphism of simplicial $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebras $\phi: A\longrightarrow B$ is* strict *if, for every $n\geq 0$, the map $\phi_n: A_n\longrightarrow B_n$ is a strict morphism of $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebras.*

Theorem 2.4.5. *Let $A,B\in\mathrm{dg}\widehat{\mathrm{Mod}}_{\mathbb{K}}$ be such that $A\otimes\Sigma N^*(\Delta^\bullet)$ and $B\otimes\Sigma N^*(\Delta^\bullet)$ are endowed with the structure of simplicial $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebras. Let $f: A\longrightarrow B$ be a surjective morphism in* $\mathrm{dg}\widehat{\mathrm{Mod}}_{\mathbb{K}}$ such that $f\otimes id: A\otimes\Sigma N^*(\Delta^\bullet)\longrightarrow B\otimes\Sigma N^*(\Delta^\bullet)$ is a strict morphism.
Then $\mathcal{MC}(f\otimes id):\mathcal{MC}(A\otimes\Sigma N^*(\Delta^\bullet))\longrightarrow\mathcal{MC}(B\otimes\Sigma N^*(\Delta^\bullet))$ is a Kan fibration.

Proof. Since the $\Gamma\Lambda\mathcal{PL}_\infty$-algebra structures are compatible with the simplicial structures on $A\otimes\Sigma N^*(\Delta^\bullet)$ and $B\otimes\Sigma N^*(\Delta^\bullet)$, we can follow the same proof as in [KW21, Proposition 6.6] to obtain the result. □

Applying this result to $B=0$ thus gives the following corollary.

Corollary 2.4.6. *For every $A\in\mathrm{dg}\widehat{\mathrm{Mod}}_{\mathbb{K}}$ such that $A\otimes\Sigma N^*(\Delta^\bullet)$ is a strict simplicial $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebra, the simplicial set $\mathcal{MC}(A\otimes\Sigma N^*(\Delta^\bullet))$ is a Kan complex.*
In particular, for every complete brace algebra A, the simplicial set $\mathcal{MC}_\bullet(A)$ is a Kan complex.

2.4.2 Connected components and homotopy groups of $\mathcal{MC}_\bullet(A)$

We are now able to compute the connected components and the homotopy groups of $\mathcal{MC}_\bullet(A)$ for a given complete brace algebra A. For this purpose, recall from [Ver23, Theorem 2.15] that any brace algebra A is endowed with the structure of a $\Gamma(\mathcal{P}re\mathcal{L}ie,-)$-algebra via the composite

$$\Gamma(\mathcal{P}re\mathcal{L}ie,A)\longrightarrow\Gamma(\mathcal{B}race,A)\xleftarrow{\simeq}\mathcal{S}(\mathcal{B}race,A)\longrightarrow A.$$

In this setting, we recall from [Ver23, Definition 2.19] the operation $\circledcirc$ defined by

$$x\circledcirc(1+y) := \sum_{n\geq 0} x\langle\underbrace{y,\dots,y}_{n}\rangle$$

for every $x \in A$ and $y \in A_0$. By [Ver23, Theorem 2.24], we have that this operation induces a group structure on the set $G = 1 + A_0$ with the product

$$(1+x) \circledcirc (1+y) = 1 + x + y + \sum_{n\geq 1} x\langle \underbrace{y,\ldots,y}_{n} \rangle.$$

This group is called the *gauge group* associated to the brace algebra A. In the following, we use the operation $\overline{\circledcirc}$ defined by

$$x \overline{\circledcirc} y := x + y + \sum_{n\geq 1} x\langle \underbrace{y,\ldots,y}_{n} \rangle,$$

for every $x \in A$ and $y \in A_0$. Note that the group $(1 + A_0, \circledcirc, 1)$ is isomorphic to the group $(A_0, \overline{\circledcirc}, 0)$. Using this identification and [Ver23, Theorem 2.29], we have an action of $(A_0, \overline{\circledcirc}, 0)$ on $\mathcal{MC}(A)$ by

$$x \cdot \tau = (\tau + x\langle \tau \rangle - d(x)) \overline{\circledcirc} x^{\overline{\circledcirc}-1}$$

for every $x \in A_0$ and $\tau \in \mathcal{MC}(A)$.

By Corollary 2.3.9, we have an obvious identification $\mathcal{MC}_0(A) = \mathcal{MC}(A)$, using the Maurer-Cartan set of a $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra (see [Ver23, Definition 2.17]). This identification is given by sending $\tau \in \mathcal{MC}(A)$ to $-\tau \otimes \Sigma \underline{0}^\vee \in \mathcal{MC}_0(A)$.

In this subsection, in order to write easier formulas, for every $\underline{x} \in N_*(\Delta^n)$, we drop the desuspension Σ^{-1} on the element $\Sigma^{-1}\underline{x} \in \Sigma^{-1}N_*(\Delta^n)$. Analogously, we drop the suspension Σ on elements of $\Sigma N^*(\Delta^n)$.

Connected components

We first compute the π_0. We begin by the following lemma.

Lemma 2.4.7. *Let $\tau_0, \tau_1 \in \mathcal{MC}(A)$. Then every element $\alpha \in \mathcal{MC}_1(A)$ such that $d_0\alpha = \tau_0$ and $d_1\alpha = \tau_1$ are written*

$$\alpha = -\tau_1 \otimes \underline{0}^\vee - \tau_0 \otimes \underline{1}^\vee - h \otimes \underline{01}^\vee$$

where $h \in A_0$ is such that

$$d(h) = \tau_0 + h\langle \tau_0 \rangle - \tau_1 \circledcirc (1 + h).$$

Proof. Let $\alpha \in \mathcal{MC}_1(A)$ be such that $d_0\alpha = \tau_0$ and $d_1\alpha = \tau_1$. We write

$$\alpha = -\tau_1 \otimes \underline{0}^\vee - \tau_0 \otimes \underline{1}^\vee - h \otimes \underline{01}^\vee$$

for some $h \in A_0$. We make explicit the Maurer-Cartan condition on α. We have

$$d(\alpha) = -d(\tau_1) \otimes \underline{0}^\vee - d(\tau_0) \otimes \underline{1}^\vee + (-d(h) + \tau_0 - \tau_1) \otimes \underline{01}^\vee.$$

Let $p \geq 1$. By formula (v) of Theorem 2.2.22, we have that

$$\begin{aligned}\alpha\{\!\{\alpha\}\!\}_p = & \sum_{p_1+p_2+p_3=p} -\tau_1 \otimes \underline{0}^\vee \{\!\{-\tau_1 \otimes \underline{0}^\vee, -\tau_0 \otimes \underline{1}^\vee, -h \otimes \underline{01}^\vee\}\!\}_{p_1,p_2,p_3} \\ & - \sum_{p_1+p_2+p_3=p} \tau_0 \otimes \underline{1}^\vee \{\!\{-\tau_1 \otimes \underline{0}^\vee, -\tau_0 \otimes \underline{1}^\vee, -h \otimes \underline{01}^\vee\}\!\}_{p_1,p_2,p_3} \\ & - \sum_{p_1+p_2+p_3=p} h \otimes \underline{01}^\vee \{\!\{-\tau_1 \otimes \underline{0}^\vee, -\tau_0 \otimes \underline{1}^\vee, -h \otimes \underline{01}^\vee\}\!\}_{p_1,p_2,p_3}.\end{aligned}$$

By the computation of ∂^1 in Corollary 2.3.9, the first sum gives non-zero elements only for the case $p_1 = 1$ and $p_2 = p_3 = 0$, and the case $p_1 = p_2 = 0$. This then gives

$$-\tau_1\langle\tau_1\rangle \otimes \underline{0}^\vee - \sum_{n\geq 1} \tau_1\langle\underbrace{h,\ldots,h}_{n}\rangle \otimes \underline{01}^\vee.$$

The second sum gives non-zero elements only for the case $p_2 = 1$ and $p_1 = p_3 = 0$. We obtain the term

$$-\tau_0\langle\tau_0\rangle \otimes \underline{1}^\vee.$$

Finally, the third sum gives non-zero elements only for the case $p_2 = 1$ and $p_1 = p_3 = 0$. We have the term

$$h\langle\tau_0\rangle \otimes \underline{01}^\vee.$$

At the end, we have

$$\sum_{p\geq 0} \alpha\{\!\{\alpha\}\!\}_p = \left(-d(h) + \tau_0 + h\langle\tau_0\rangle - \sum_{n\geq 0} \tau_1\langle\underbrace{h,\ldots,h}_{n}\rangle\right) \otimes \underline{01}^\vee.$$

Then, the Maurer-Cartan condition on α is equivalent to the equation

$$d(h) = \tau_0 + h\langle\tau_0\rangle - \tau_1 \circledcirc (1+h)$$

which proves the lemma. $\square$

Recall that the Deligne groupoid associated to A is the category formed by Maurer-Cartan elements with as morphisms the elements of the gauge group (see [Ver23, Proposition-Definition 2.22]).

Theorem 2.4.8. *Let A be a complete brace algebra. We have a bijection*

$$\pi_0(\mathcal{MC}_\bullet(A)) \simeq \pi_0\mathrm{Deligne}(A),$$

where we denote by $\pi_0\mathrm{Deligne}(A)$ *the set of objects in* $\mathrm{Deligne}(A)$ *up to isomorphisms.*

Proof. Recall that

$$\pi_0(\mathcal{MC}_\bullet(A)) = \mathcal{MC}_0(A)/\sim,$$

where $\sim$ denotes the homotopy relation in sSet. Consider the projection f of $\mathcal{MC}_0(A)$ on $\mathcal{MC}(A)/G$, where G denotes the gauge group of the $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$-algebra A. Let $\tau_0, \tau_1 \in \mathcal{MC}(A)$. By Lemma 2.4.7, the elements $-\tau_1 \otimes \underline{0}^\vee$ and $-\tau_0 \otimes \underline{1}^\vee$ are homotopic in $\mathcal{MC}_0(A)$ if and only if there exists $h \in A_0$ such that $h \cdot \tau_0 = \tau_1$, which proves that f induces a bijection $\overline{f} : \pi_0(\mathcal{MC}_\bullet(A)) \longrightarrow \pi_0\mathrm{Deligne}(A)$. $\square$

The group $\pi_1(\mathcal{MC}_\bullet(A), \tau)$

We now compute $\pi_1(\mathcal{MC}_\bullet(A), \tau)$ for a given $\tau \in \mathcal{MC}(A)$. Let $\mathrm{Aut}_{\mathrm{Deligne}(A)}(\tau) = \{h \in A_0 \mid d(h) = \tau + h\langle\tau\rangle - \tau \circledcirc (1+h)\}$. We have the following lemma.

Lemma 2.4.9. *Let A be a complete brace algebra and $\tau \in \mathcal{MC}(A)$. For every $h, h' \in \mathrm{Aut}_{\mathrm{Deligne}(A)}(\tau)$, we write $h \sim_\tau h'$ if there exists $\psi \in A_1$ such that*

$$h - h' = d(\psi) + \psi\langle\tau\rangle + \sum_{p,q \geq 0} \tau\langle\underbrace{h, \dots, h}_{p}, \psi, \underbrace{h', \dots, h'}_{q}\rangle.$$

Then $\sim_\tau$ is an equivalence relation on the set $\mathrm{Aut}_{\mathrm{Deligne}(A)}(\tau)$. Moreover, the circular product $\circledcirc$ is compatible with $\sim_\tau$, so that the triple $(\mathrm{Aut}_{\mathrm{Deligne}(A)}(\tau)/\sim_\tau, \overline{\circledcirc}, 0)$ is a group.

Proof. The relation $\sim_\tau$ is reflexive (just take $\psi = 0$ so that $h \sim_\tau h$ for all $h \in \mathrm{Aut}_{\mathrm{Deligne}(A)}(\tau)$). We prove that this relation is transitive. Let $h, h', h'' \in A_0$ be such that $h \sim_\tau h'$ and $h' \sim_\tau h''$. Then there exist $\psi, \psi' \in A_1$ such that

$$h - h' = d(\psi) + \psi\langle\tau\rangle + \sum_{p,q \geq 0} \tau\langle\underbrace{h, \dots, h}_{q}, \psi, \underbrace{h', \dots, h'}_{q}\rangle; \tag{2.4.1}$$

$$h' - h'' = d(\psi') + \psi'\langle\tau\rangle + \sum_{p,q \geq 0} \tau\langle\underbrace{h', \dots, h'}_{q}, \psi', \underbrace{h'', \dots, h''}_{q}\rangle. \tag{2.4.2}$$

We set $\psi'' := \psi + \psi' + \sum_{p,q,r \geq 0} \tau\langle\underbrace{h, \dots, h}_{p}, \psi, \underbrace{h', \dots, h'}_{q}, \psi', \underbrace{h'', \dots, h''}_{r}\rangle$, and prove that

$$h - h'' = d(\psi'') + \psi''\langle\tau\rangle + \sum_{p,q \geq 0} \tau\langle\underbrace{h, \dots, h}_{p}, \psi'', \underbrace{h'', \dots, h''}_{q}\rangle.$$

Let us analyze the right hand-side. We analyze the terms given by $d(\psi'')$ and compare it with the others given either by $\psi''\langle\tau\rangle$ or by the terms of the form $\tau\langle h, \dots, h, \psi'', h'', \dots, h''\rangle$. We first have

$$\begin{aligned} d(\psi) + d(\psi') &= h - h'' - \psi\langle\tau\rangle - \psi'\langle\tau\rangle \\ &\quad - \sum_{p,q \geq 0} \tau\langle\underbrace{h, \dots, h}_{p}, \psi, \underbrace{h', \dots, h'}_{q}\rangle - \sum_{q,r \geq 0} \tau\langle\underbrace{h', \dots, h'}_{q}, \psi', \underbrace{h'', \dots, h''}_{r}\rangle. \end{aligned}$$

We now differentiate the sum which occurs in the definition of ψ''. By the Leibniz rule in the brace algebra A, and by applying the differential on $\tau \in \mathcal{MC}(A)$, we, in particular, obtain the sum

$$- \sum_{p,q,r \geq 0} \tau\langle\tau\rangle\langle\underbrace{h, \dots, h}_{p}, \psi, \underbrace{h', \dots, h'}_{q}, \psi', \underbrace{h'', \dots, h''}_{r}\rangle.$$

This can be computed by using the brace algebra structure of A:

$$
\begin{aligned}
-\sum_{p,q,r\geq 0} \tau\langle\tau\rangle\langle\underbrace{h,\dots,h}_{p},\psi,\underbrace{h',\dots,h'}_{q},\psi',\underbrace{h'',\dots,h''}_{r}\rangle =\\
-\sum_{p_1,p_2,q,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p_1},\tau\circledcirc(1+h),\underbrace{h,\dots,h}_{p_2},\psi,\underbrace{h',\dots,h'}_{q},\psi',\underbrace{h'',\dots,h''}_{r}\rangle\\
-\sum_{p,q,r,s,t\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},\tau\langle\underbrace{h,\dots,h}_{s},\psi,\underbrace{h',\dots,h'}_{t}\rangle,\underbrace{h',\dots,h'}_{q},\psi',\underbrace{h'',\dots,h''}_{r}\rangle\\
+\sum_{p,q_1,q_2,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},\psi,\underbrace{h',\dots,h'}_{q_1},\tau\circledcirc(1+h'),\underbrace{h',\dots,h'}_{q_2},\psi',\underbrace{h'',\dots,h''}_{r}\rangle\\
+\sum_{p,q,r,s,t\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},\psi,\underbrace{h',\dots,h'}_{q},\tau\langle\underbrace{h',\dots,h'}_{s},\psi',\underbrace{h'',\dots,h''}_{t}\rangle,\underbrace{h'',\dots,h''}_{r}\rangle\\
-\sum_{p,q,r_1,r_2\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},\psi,\underbrace{h',\dots,h'}_{r},\psi',\underbrace{h'',\dots,h''}_{r_1},\tau\circledcirc(1+h''),\underbrace{h'',\dots,h''}_{r_2}\rangle\\
-\sum_{p,q,r,s,t\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},\tau\langle\underbrace{h,\dots,h}_{q},\psi,\underbrace{h',\dots,h'}_{s},\psi',\underbrace{h'',\dots,h''}_{t}\rangle,\underbrace{h'',\dots,h''}_{r}\rangle.
\end{aligned}
$$

The remaining terms obtained by the Leibniz rule in the sum occurring in the definition of ψ'' are

$$
\begin{aligned}
&-\sum_{p_1,p_2,q,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p_1},d(h),\underbrace{h,\dots,h}_{p_2},\psi,\underbrace{h',\dots,h'}_{q},\psi',\underbrace{h'',\dots,h''}_{r}\rangle\\
&-\sum_{p,q,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},d(\psi),\underbrace{h',\dots,h'}_{q},\psi',\underbrace{h'',\dots,h''}_{r}\rangle\\
&+\sum_{p,q_1,q_2,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},\psi,\underbrace{h',\dots,h'}_{q_1},d(h'),\underbrace{h',\dots,h'}_{q_2},\psi',\underbrace{h'',\dots,h''}_{r}\rangle\\
&+\sum_{p,q,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},\psi,\underbrace{h',\dots,h'}_{q},d(\psi'),\underbrace{h'',\dots,h''}_{r}\rangle\\
&-\sum_{p,q,r_1,r_2\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},\psi,\underbrace{h',\dots,h'}_{r},\psi',\underbrace{h'',\dots,h''}_{r_1},d(h''),\underbrace{h'',\dots,h''}_{r_2}\rangle.
\end{aligned}
$$

By using equations (1) and (2), the definition of ψ'' and that $h,h',h''\in \mathrm{Aut}_{\mathrm{Deligne}(A)}(\tau)$, we obtain

$$
\begin{aligned}
d(\psi'') = h - h'' - \psi'\langle\tau\rangle - \psi'\langle\tau\rangle - \sum_{p,q\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},\psi,\underbrace{h',\dots,h'}_{q}\rangle \\
- \sum_{q,r\geq 0} \tau\langle\underbrace{h',\dots,h'}_{q},\psi',\underbrace{h'',\dots,h''}_{r}\rangle \\
- \sum_{p_1,p_2,q,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p_1},\tau + h\langle\tau\rangle,\underbrace{h,\dots,h}_{p_2},\psi,\underbrace{h',\dots,h'}_{q},\psi',\underbrace{h'',\dots,h''}_{r}\rangle \\
- \sum_{p,q,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},h - h' - \psi\langle\tau\rangle,\underbrace{h',\dots,h'}_{q},\psi',\underbrace{h'',\dots,h''}_{r}\rangle \\
+ \sum_{p,q_1,q_2,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},\psi,\underbrace{h',\dots,h'}_{q_1},\tau + h'\langle\tau\rangle,\underbrace{h',\dots,h'}_{q_2},\psi',\underbrace{h'',\dots,h''}_{r}\rangle \\
+ \sum_{p,q,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},\psi,\underbrace{h',\dots,h'}_{q},h' - h'' - \psi'\langle\tau\rangle,\underbrace{h'',\dots,h''}_{r}\rangle \\
- \sum_{p,q,r_1,r_2\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},\psi,\underbrace{h',\dots,h'}_{r},\psi',\underbrace{h'',\dots,h''}_{r_1},\tau + h''\langle\tau\rangle,\underbrace{h'',\dots,h''}_{r_2}\rangle \\
- \sum_{p,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},\psi'' - \psi - \psi',\underbrace{h'',\dots,h''}_{r}\rangle.
\end{aligned}
$$

Now, by some variable substitutions, note that we have the identities

$$
\begin{aligned}
&\sum_{p,q,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},h - h',\underbrace{h',\dots,h'}_{q},\psi',\underbrace{h'',\dots,h''}_{r}\rangle \\
&\qquad = \sum_{p,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},\psi',\underbrace{h'',\dots,h''}_{r}\rangle - \sum_{q,r\geq 0} \tau\langle\underbrace{h',\dots,h'}_{q},\psi',\underbrace{h'',\dots,h''}_{r}\rangle;
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{p,q,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},\psi,\underbrace{h',\dots,h'}_{q},h' - h'',\underbrace{h'',\dots,h''}_{r}\rangle \\
&\qquad = \sum_{p,q\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},\psi,\underbrace{h',\dots,h'}_{q}\rangle - \sum_{p,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},\psi,\underbrace{h'',\dots,h''}_{r}\rangle.
\end{aligned}
$$

This finally gives

$$
\begin{aligned}
d(\psi'') = h - h'' - \psi'\langle\tau\rangle - \psi'\langle\tau\rangle \\
- \sum_{p_1,p_2,q,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p_1}, \tau + h\langle\tau\rangle, \underbrace{h,\dots,h}_{p_2}, \psi, \underbrace{h',\dots,h'}_{q}, \psi', \underbrace{h'',\dots,h''}_{r}\rangle \\
+ \sum_{p,q,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p}, \psi\langle\tau\rangle, \underbrace{h',\dots,h'}_{q}, \psi', \underbrace{h'',\dots,h''}_{r}\rangle \\
+ \sum_{p,q_1,q_2,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p}, \psi, \underbrace{h',\dots,h'}_{q_1}, \tau + h'\langle\tau\rangle, \underbrace{h',\dots,h'}_{q_2}, \psi', \underbrace{h'',\dots,h''}_{r}\rangle \\
- \sum_{p,q,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p}, \psi, \underbrace{h',\dots,h'}_{q}, \psi'\langle\tau\rangle, \underbrace{h'',\dots,h''}_{r}\rangle \\
- \sum_{p,q,r_1,r_2\geq 0} \tau\langle\underbrace{h,\dots,h}_{p}, \psi, \underbrace{h',\dots,h'}_{r}, \psi', \underbrace{h'',\dots,h''}_{r_1}, \tau + h''\langle\tau\rangle, \underbrace{h'',\dots,h''}_{r_2}\rangle \\
- \sum_{p,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p}, \psi'', \underbrace{h'',\dots,h''}_{r}\rangle.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\psi''\langle\tau\rangle = \psi\langle\tau\rangle + \psi'\langle\tau\rangle + \sum_{p_1,p_2,q,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p_1}, \tau + h\langle\tau\rangle, \underbrace{h,\dots,h}_{p_2}, \psi, \underbrace{h',\dots,h'}_{q}, \psi', \underbrace{h'',\dots,h''}_{r}\rangle \\
- \sum_{p,q,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p}, \psi\langle\tau\rangle, \underbrace{h',\dots,h'}_{q}, \psi', \underbrace{h'',\dots,h''}_{r}\rangle \\
- \sum_{p,q_1,q_2,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p}, \psi, \underbrace{h',\dots,h'}_{q_1}, \tau + h'\langle\tau\rangle, \underbrace{h',\dots,h'}_{q_2}, \psi', \underbrace{h'',\dots,h''}_{r}\rangle \\
+ \sum_{p,q,r\geq 0} \tau\langle\underbrace{h,\dots,h}_{p}, \psi, \underbrace{h',\dots,h'}_{q}, \psi'\langle\tau\rangle, \underbrace{h'',\dots,h''}_{r}\rangle \\
+ \sum_{p,q,r_1,r_2\geq 0} \tau\langle\underbrace{h,\dots,h}_{p}, \psi, \underbrace{h',\dots,h'}_{q}, \psi', \underbrace{h'',\dots,h''}_{r_1}, \tau + h''\langle\tau\rangle, \underbrace{h'',\dots,h''}_{r_2}\rangle.
\end{aligned}
$$

At the end, we obtain

$$
d(\psi'') + \psi''\langle\tau\rangle = h - h'' - \sum_{p,q\geq 0} \tau\langle\underbrace{h,\dots,h}_{p}, \psi'', \underbrace{h'',\dots,h''}_{q}\rangle
$$

which proves that $h \sim_\tau h''$.

We now prove that if $h \sim_\tau h'$, then $h' \sim_\tau h$. We use the previous construction. More precisely, let $\psi \in A_1$ be such that

$$
h - h' = d(\psi) + \psi\langle\tau\rangle + \sum_{p,q\geq 0} \tau\langle\underbrace{h,\dots,h}_{p}, \psi, \underbrace{h',\dots,h'}_{q}\rangle.
$$

We search some element ψ' such that the associated ψ'' previously constructed for the transitivity is 0. We set $\psi'_0 = -\psi$ and, for all $n \geq 0$,

$$\psi'_{n+1} = -\psi - \sum_{p,q,r\geq 0} \tau\langle \underbrace{h,\ldots,h}_{q}, \psi, \underbrace{h',\ldots,h'}_{q}, \psi'_n, \underbrace{h,\ldots,h}_{r}\rangle.$$

We obtain a Cauchy sequence $(\psi'_n)_n$. Let ψ' be its limit, which satisfies

$$\psi' = -\psi - \sum_{p,q,r\geq 0} \tau\langle \underbrace{h,\ldots,h}_{p}, \psi, \underbrace{h',\ldots,h'}_{q}, \psi', \underbrace{h,\ldots,h}_{r}\rangle.$$

By the same computations as for the proof of the transitivity, we can check that ψ' satisfies the equation

$$h' - h = d(\psi') + \psi'\langle\tau\rangle + \sum_{p,q\geq 0} \tau\langle \underbrace{h',\ldots,h'}_{p}, \psi', \underbrace{h,\ldots,h}_{q}\rangle,$$

which proves that $h' \sim_\tau h$.

We thus have proved that $\sim_\tau$ is an equivalence relation. We now prove that the circular product $\circledcirc$ is compatible with $\sim_\tau$. Let $h, h_1, h_2 \in A_0$ be such that $h_1 \sim_\tau h_2$. Let $\psi \in A_1$ be such that

$$h_1 - h_2 = d(\psi) + \psi\langle\tau\rangle + \sum_{p,q\geq 0} \tau\langle \underbrace{h_1,\ldots,h_1}_{p}, \psi, \underbrace{h_2,\ldots,h_2}_{q}\rangle. \tag{2.4.3}$$

We prove first that $h_1\overline{\circledcirc}h \sim_\tau h_2\overline{\circledcirc}h$. Let $\psi' := \psi \circledcirc (1+h)$. We need to show that

$$h_1\overline{\circledcirc}h - h_2\overline{\circledcirc}h = d(\psi') + \psi'\langle\tau\rangle + \sum_{p,q\geq 0} \tau\langle \underbrace{h_1\overline{\circledcirc}h,\ldots,h_1\overline{\circledcirc}h}_{p}, \psi', \underbrace{h_2\overline{\circledcirc}h,\ldots,h_2\overline{\circledcirc}h}_{q}\rangle.$$

We first compute $d(\psi')$. From [Ver23, Lemma 2.28], we have

$$d(\psi') = d(\psi) \circledcirc (1+h) - \psi \circledcirc (1+h; d(h)),$$

where we have set, following [Ver23, Definition 2.27] in the case of a complete brace algebra,

$$a \circledcirc (1+b;c) := \sum_{n\geq 0}\sum_{k=0}^{n} a\langle b,\ldots,b,\underset{k}{c},b,\ldots,b\rangle.$$

for every $a \in A$ and $b, c \in A_0$. We have

$$d(\psi)\circledcirc(1+h) = h_1\overline{\circledcirc}h - h_2\overline{\circledcirc}h - \psi\langle\tau\rangle\circledcirc(1+h) - \sum_{p,q\geq 0} \tau\langle \underbrace{h_1,\ldots,h_1}_{p}, \psi, \underbrace{h_2,\ldots,h_2}_{q}\rangle\circledcirc(1+h).$$

By the second formula of [Ver23, Lemma 2.28], we have

$$\psi\langle\tau\rangle \circledcirc (1+h) = \psi \circledcirc (1+h; \tau\circledcirc(1+h)).$$

Finally, we have

$$\sum_{p,q\geq 0} \tau\langle \underbrace{h_1,\dots,h_1}_{p}, \psi, \underbrace{h_2,\dots,h_2}_{q}\rangle \circledcirc (1+h)$$
$$= \sum_{p,q\geq 0} \tau\langle \underbrace{h_1\overline{\circledcirc}h,\dots,h_1\overline{\circledcirc}h}_{p}, \psi\circledcirc(1+h), \underbrace{h_2\overline{\circledcirc}h,\dots,h_2\overline{\circledcirc}h}_{q}\rangle.$$

We thus have

$$d(\psi') = h_1\overline{\circledcirc}h - h_2\overline{\circledcirc}h - \psi\circledcirc(1+h;\tau\circledcirc(1+h))$$
$$- \sum_{p,q\geq 0} \tau\langle \underbrace{h_1\overline{\circledcirc}h,\dots,h_1\overline{\circledcirc}h}_{p}, \psi', \underbrace{h_2\overline{\circledcirc}h,\dots,h_2\overline{\circledcirc}h}_{q}\rangle - \psi\circledcirc(1+h;d(h)).$$

Since we have, by [Ver23, Lemma 2.28], that

$$\psi'\langle\tau\rangle = \psi\circledcirc(1+h;\tau+h\langle\tau\rangle).$$

we obtain at the end

$$d(\psi') + \psi'\langle\tau\rangle + \sum_{p,q\geq 0} \tau\langle \underbrace{h_1\overline{\circledcirc}h,\dots,h_1\overline{\circledcirc}h}_{p}, \psi', \underbrace{h_2\overline{\circledcirc}h,\dots,h_2\overline{\circledcirc}h}_{q}\rangle = h_1\overline{\circledcirc}h - h_2\overline{\circledcirc}h$$

which proves that $h_1\overline{\circledcirc}h \sim_\tau h_2\overline{\circledcirc}h$.

We now prove that $h\overline{\circledcirc}h_1 \sim_\tau h\overline{\circledcirc}h_2$. Let $\psi' = \psi + \sum_{p,q\geq 0} h\langle \underbrace{h_1,\dots,h_1}_{p}, \psi, \underbrace{h_2,\dots,h_2}_{q}\rangle$. We show that

$$h\overline{\circledcirc}h_1 - h\overline{\circledcirc}h_2 = d(\psi') + \psi'\langle\tau\rangle + \sum_{p,q\geq 0} \tau\langle \underbrace{h\overline{\circledcirc}h_1,\dots,h\overline{\circledcirc}h_1}_{p}, \psi', \underbrace{h\overline{\circledcirc}h_2,\dots,h\overline{\circledcirc}h_2}_{q}\rangle.$$

We first compute the sum $\sum_{p,q\geq 0} d(h)\langle \underbrace{h_1,\dots,h_1}_{p}, \psi, \underbrace{h_2,\dots,h_2}_{q}\rangle$. We use that $d(h) = \tau + h\langle\tau\rangle - \tau\circledcirc(1+h)$ to get

$$\begin{aligned}\sum_{p,q\geq 0} d(h)\langle \underbrace{h_1,\ldots,h_1}_{p},\psi,\underbrace{h_2,\ldots,h_2}_{q}\rangle &= \sum_{p,q\geq 0}\tau\langle \underbrace{h_1,\ldots,h_1}_{p},\psi,\underbrace{h_2,\ldots,h_2}_{q}\rangle\\
&+\sum_{p_1,p_2,q\geq 0} h\langle \underbrace{h_1,\ldots,h_1}_{p_1},\tau\circledcirc(1+h_1),\underbrace{h_1,\ldots,h_1}_{p_2},\psi,\underbrace{h_2,\ldots,h_2}_{q}\rangle\\
&+\sum_{p,q,s,t\geq 0} h\langle \underbrace{h_1,\ldots,h_1}_{p},\tau\langle\underbrace{h_1,\ldots,h_1}_{s},\psi,\underbrace{h_2,\ldots,h_2}_{t}\rangle,\underbrace{h_2,\ldots,h_2}_{q}\rangle\\
&-\sum_{p,q_1,q_2\geq 0} h\langle \underbrace{h_1,\ldots,h_1}_{p},\psi,\underbrace{h_2,\ldots,h_2}_{q_1},\tau\circledcirc(1+h_2),\underbrace{h_2,\ldots,h_2}_{q_1}\rangle\\
&-\sum_{p,q,s,t\geq 0}\tau\langle \underbrace{h\overline{\circledcirc}h_1,\ldots,h\overline{\circledcirc}h_1}_{p},\psi+h\langle\underbrace{h_1,\ldots,h_1}_{s},\psi,\underbrace{h_2,\ldots,h_2}_{t}\rangle,\underbrace{h\overline{\circledcirc}h_2,\ldots,h\overline{\circledcirc}h_2}_{q}\rangle.\end{aligned}$$

Using the Leibniz rule, we obtain

$$\begin{aligned}d(\psi') = d(\psi) &+ \sum_{p,q\geq 0}\tau\langle \underbrace{h_1,\ldots,h_1}_{p},\psi,\underbrace{h_2,\ldots,h_2}_{q}\rangle\\
&+\sum_{p_1,p_2,q\geq 0} h\langle \underbrace{h_1,\ldots,h_1}_{p_1},\tau\circledcirc(1+h_1),\underbrace{h_1,\ldots,h_1}_{p_2},\psi,\underbrace{h_2,\ldots,h_2}_{q}\rangle\\
&+\sum_{p,q,s,t\geq 0} h\langle \underbrace{h_1,\ldots,h_1}_{p},\tau\langle\underbrace{h_1,\ldots,h_1}_{s},\psi,\underbrace{h_2,\ldots,h_2}_{t}\rangle,\underbrace{h_2,\ldots,h_2}_{q}\rangle\\
&-\sum_{p,q_1,q_2\geq 0} h\langle \underbrace{h_1,\ldots,h_1}_{p},\psi,\underbrace{h_2,\ldots,h_2}_{q_1},\tau\circledcirc(1+h_2),\underbrace{h_2,\ldots,h_2}_{q_1}\rangle\\
&-\sum_{p,q,s,t\geq 0}\tau\langle \underbrace{h\overline{\circledcirc}h_1,\ldots,h\overline{\circledcirc}h_1}_{p},\psi+h\langle\underbrace{h_1,\ldots,h_1}_{s},\psi,\underbrace{h_2,\ldots,h_2}_{t}\rangle,\underbrace{h\overline{\circledcirc}h_2,\ldots,h\overline{\circledcirc}h_2}_{q}\rangle\\
&+\sum_{p_1,p_2,q\geq 0} h\langle \underbrace{h_1,\ldots,h_1}_{p_1},d(h_1),\underbrace{h_1,\ldots,h_1}_{p_2},\psi,\underbrace{h_2,\ldots,h_2}_{q}\rangle\\
&+\sum_{p,q\geq 0} h\langle \underbrace{h_1,\ldots,h_1}_{p},d(\psi),\underbrace{h_2,\ldots,h_2}_{q}\rangle\\
&-\sum_{p,q_1,q_2\geq 0} h\langle \underbrace{h_1,\ldots,h_1}_{p},\psi,\underbrace{h_2,\ldots,h_2}_{q_1},d(h_2),\underbrace{h_2,\ldots,h_2}_{q_1}\rangle.\end{aligned}$$

Using equation (3) and that $h_1, h_2 \in \mathrm{Aut}_{\mathrm{Deligne}(A)}(\tau)$, we deduce

$$\begin{aligned}
d(\psi') = h_1 - h_2 - \psi\langle\tau\rangle + \sum_{p_1,p_2,q\geq 0} & h\langle\underbrace{h_1,\ldots,h_1}_{p_1},\tau + h_1\langle\tau\rangle,\underbrace{h_1,\ldots,h_1}_{p_2},\psi,\underbrace{h_2,\ldots,h_2}_{q}\rangle \\
& - \sum_{p,q_1,q_2\geq 0} h\langle\underbrace{h_1,\ldots,h_1}_{p},\psi,\underbrace{h_2,\ldots,h_2}_{q_1},\tau + h_2\langle\tau\rangle,\underbrace{h_2,\ldots,h_2}_{q_1}\rangle \\
& + \sum_{p,q\geq 0} h\langle\underbrace{h_1,\ldots,h_1}_{p},h_1 - h_2 - \psi\langle\tau\rangle,\underbrace{h_2,\ldots,h_2}_{q}\rangle \\
& - \sum_{p,q,\geq 0} \tau\langle\underbrace{h\circledcirc h_1,\ldots,h\circledcirc h_1}_{p},\psi',\underbrace{h\circledcirc h_2,\ldots,h\circledcirc h_2}_{q}\rangle.
\end{aligned}$$

Since we have

$$\sum_{p,q\geq 0} h\langle\underbrace{h_1,\ldots,h_1}_{p},h_1 - h_2,\underbrace{h_2,\ldots,h_2}_{q}\rangle = \sum_{p\geq 0} h\langle\underbrace{h_1,\ldots,h_1}_{p}\rangle - \sum_{r\geq 0} h\langle\underbrace{h_2,\ldots,h_2}_{r}\rangle,$$

we finally obtain

$$\begin{aligned}
d(\psi') = h\overline{\circledcirc}h_1 - h\overline{\circledcirc}h_2 - \psi\langle\tau\rangle + \sum_{p_1,p_2,q\geq 0} & h\langle\underbrace{h_1,\ldots,h_1}_{p_1},\tau + h_1\langle\tau\rangle,\underbrace{h_1,\ldots,h_1}_{p_2},\psi,\underbrace{h_2,\ldots,h_2}_{q}\rangle \\
& - \sum_{p,q_1,q_2\geq 0} h\langle\underbrace{h_1,\ldots,h_1}_{p},\psi,\underbrace{h_2,\ldots,h_2}_{q_1},\tau + h_2\langle\tau\rangle,\underbrace{h_2,\ldots,h_2}_{q_1}\rangle \\
& - \sum_{p,q\geq 0} h\langle\underbrace{h_1,\ldots,h_1}_{p},\psi\langle\tau\rangle,\underbrace{h_2,\ldots,h_2}_{q}\rangle \\
& - \sum_{p,q,\geq 0} \tau\langle\underbrace{h\overline{\circledcirc}h_1,\ldots,h\overline{\circledcirc}h_1}_{p},\psi',\underbrace{h\overline{\circledcirc}h_2,\ldots,h\overline{\circledcirc}h_2}_{q}\rangle.
\end{aligned}$$

We also have

$$\begin{aligned}
\psi'\langle\tau\rangle = \psi\langle\tau\rangle - \sum_{p_1,p_2,q\geq 0} & h\langle\underbrace{h_1,\ldots,h_1}_{p_1},\tau + h_1\langle\tau\rangle,\underbrace{h_1,\ldots,h_1}_{p_2},\psi,\underbrace{h_2,\ldots,h_2}_{q}\rangle \\
& + \sum_{p,q\geq 0} h\langle\underbrace{h_1,\ldots,h_1}_{p},\psi\langle\tau\rangle,\underbrace{h_2,\ldots,h_2}_{q}\rangle \\
& + \sum_{p,q_1,q_2\geq 0} h\langle\underbrace{h_1,\ldots,h_1}_{p},\psi,\underbrace{h_2,\ldots,h_2}_{q_1},\tau + h_2\langle\tau\rangle,\underbrace{h_2,\ldots,h_2}_{q_2}\rangle.
\end{aligned}$$

At the end, we obtain

$$d(\psi') + \psi'\langle\tau\rangle = h\overline{\circledcirc}h_1 - h\overline{\circledcirc}h_2 - \sum_{p,q,\geq 0} \tau\langle\underbrace{h\overline{\circledcirc}h_1,\ldots,h\overline{\circledcirc}h_1}_{p},\psi',\underbrace{h\overline{\circledcirc}h_2,\ldots,h\overline{\circledcirc}h_2}_{q}\rangle$$

which proves that $h\overline{\circledcirc}h_1 \sim_\tau h\overline{\circledcirc}h_2$. The lemma is proved. $\square$

Theorem 2.4.10. *Let A be a complete brace algebra and $\tau \in \mathcal{MC}(A)$. Then*

$$\pi_1(\mathcal{MC}_\bullet(A),\tau) \simeq \mathrm{Aut}_{\mathrm{Deligne}(A)}(\tau)/\sim_\tau .$$

Proof. Recall that $\mathrm{Aut}_{\mathrm{Deligne(A)}}(\tau) = \{h \in A_0 \mid d(h) = \tau + h\langle\tau\rangle - \tau \circledcirc (1+h)\}$. Let $h \in A_0$. By Lemma 2.4.7, we have that $h \in \mathrm{Aut}_{\mathrm{Deligne(A)}}(\tau)$ if and only if

$$-\tau \otimes (\underline{0}^\vee + \underline{1}^\vee) - h \otimes \underline{01}^\vee \in \mathcal{MC}(A \otimes \Sigma N^*(\Delta^1)).$$

We thus have a bijection

$$f : \mathrm{Aut}_{\mathrm{Deligne(A)}}(\tau) \longrightarrow \mathcal{MC}_1(A)_\tau$$

where we denote by $\mathcal{MC}_1(A)_\tau$ the subset of $\mathcal{MC}_1(A)$ given by elements whose 0 and 1 vertices are given by τ. Consider now $h, h' \in A_0$ such that

$$d(h) = \tau + h\langle\tau\rangle - \tau \circledcirc (1+h);$$

$$d(h') = \tau + h'\langle\tau\rangle - \tau \circledcirc (1+h').$$

Let $\xi \in \mathcal{MC}_2(A)$ be such that $d_1\xi = f(h)$ and $d_2\xi = f(h')$. We write ξ as

$$\xi = -\tau \otimes (\underline{0}^\vee + \underline{1}^\vee + \underline{2}^\vee) - h' \otimes \underline{01}^\vee - h \otimes \underline{02}^\vee + \psi \otimes \underline{012}^\vee$$

for some $\psi \in A_1$. We make precise the Maurer-Cartan condition on ξ. We first have

$$d(\xi) = -d(\tau) \otimes (\underline{0}^\vee + \underline{1}^\vee + \underline{2}^\vee) - d(h') \otimes \underline{01}^\vee - d(h) \otimes \underline{02}^\vee + (d(\psi) - h + h') \otimes \underline{012}^\vee.$$

By Lemma 2.3.8, we have

$$\begin{aligned}\xi\{\!\{\xi\}\!\}_1 = -\tau\langle\tau\rangle \otimes (\underline{0}^\vee + \underline{1}^\vee + \underline{2}^\vee) - (\tau\langle h'\rangle - h'\langle\tau\rangle) \otimes \underline{01}^\vee \\ - (\tau\langle h\rangle + h\langle\tau\rangle) \otimes \underline{02}^\vee + (\tau\langle\psi\rangle + \psi\langle\tau\rangle) \otimes \underline{012}^\vee.\end{aligned}$$

Let $r \geq 2$. From the computations of ∂^1 and ∂^2 in Corollary 2.3.9, we deduce

$$\xi\{\!\{\xi\}\!\}_r = -\tau\langle\underbrace{h,\dots,h}_{r}\rangle \otimes \underline{01}^\vee - \tau\langle\underbrace{h',\dots,h'}_{r}\rangle \otimes \underline{02}^\vee + \sum_{p+q=r-1} \tau\langle\underbrace{h,\dots,h}_{p},\psi,\underbrace{h',\dots,h'}_{q}\rangle \otimes \underline{012}^\vee.$$

We thus have proved that ξ is a Maurer-Cartan element if and only if

$$h - h' = d(\psi) + \psi\langle\tau\rangle + \sum_{p,q\geq 0} \tau\langle\underbrace{h,\dots,h}_{p},\psi,\underbrace{h',\dots,h'}_{q}\rangle.$$

Equivalently, we have that $[f(h)] = [f(h')]$ if and only if $h \sim_\tau h'$. We thus have a well defined bijection

$$\overline{f} : \mathrm{Aut}_{\mathrm{Deligne(A)}}(\tau)/\sim_\tau \longrightarrow \pi_1(\mathcal{MC}_\bullet(A),\tau),$$

We now check that $\overline{f}$ is compatible with the group structures. Let $h, h' \in A_0$ be such that $\alpha = -\tau \otimes (\underline{0}^\vee + \underline{1}^\vee) - h \otimes \underline{01}^\vee$ and $\alpha' = -\tau \otimes (\underline{0}^\vee + \underline{1}^\vee) - h' \otimes \underline{01}^\vee$ are Maurer-Cartan elements in $\mathcal{MC}(A \otimes \Sigma N^*(\Delta^1))$. As we have seen before, by Lemma 2.4.7, it is equivalent to ask

$$d(\mu) = \tau + h\langle\tau\rangle - \tau \circledcirc (1 + h);$$
$$d(h') = \tau + h'\langle\tau\rangle - \tau \circledcirc (1 + h').$$

By Corollary 2.3.9, we see that

$$-\tau \otimes (\underline{0}^\vee + \underline{1}^\vee + \underline{2}^\vee) - h' \otimes \underline{12}^\vee - (h\overline{\circledcirc}h') \otimes \underline{02}^\vee - h \otimes \underline{01}^\vee \in \mathcal{MC}(A \otimes \Sigma N^*(\Delta^2)).$$

We then have

$$[\alpha] \cdot [\alpha'] = [-\tau \otimes (\underline{0}^\vee + \underline{1}^\vee) - (h\overline{\circledcirc}h') \otimes \underline{01}^\vee]$$

in $\pi_1(\mathcal{MC}_\bullet(A), \tau)$, which gives

$$\overline{f}([\alpha] \cdot [\alpha']) = h\overline{\circledcirc}h' = \overline{f}([\alpha]) \circledcirc \overline{f}([\alpha']),$$

showing that $\overline{f}$ is an isomorphism of groups. $\square$

The group $\pi_2(\mathcal{MC}_\bullet(A), \tau)$

We now compute the group $\pi_2(\mathcal{MC}_\bullet(A), \tau)$. We begin by general lemmas that will also be useful for the computations of $\pi_n(\mathcal{MC}_\bullet(A), \tau)$ for $n \geq 3$.

Lemma 2.4.11. *Let T be a canonical tree with $|T| \geq 3$, and $n \geq 1$. If the first branch of T has only one vertex, then there is no element of the form $\underline{0}, \ldots, \underline{n} \in \Sigma^{-1}N_*(\Delta^n)$ among the non-root vertices in the tensor products produced by $T \otimes \Lambda\mu_T(\underline{0\cdots n}) \in \mathcal{B}race^c(\Sigma^{-1}N_*(\Delta^n))$.*

Proof. For every finite set E and $k \in E$, we denote by $\pi_{\{k\}} : \chi(E) \longrightarrow \chi(E \setminus \{k\})$ the morphism which forgets the element k. If a surjection has multiple occurrences of the element k, then its image by $\pi_{\{k\}}$ is 0 by convention. Note that if A and B are disjoint finite sets, then for every $u \in \chi(A), v \in \chi(B)$, we have $\pi_{\{k\}}(u \cdot v) = \pi_{\{k\}}(u) \cdot \pi_{\{k\}}(v)$ in $\chi(A \sqcup B)$.

Let T be a canonical tree with $|T| \geq 3$. By Lemma 2.3.5, there exists $u_T \in \chi(V_T \setminus \{1\})$ such that

$$TR(\mu_T) = 12 \cdot u_T.$$

We write uniquely u_T as

$$u_T = \sum_{i=1}^{m_T} \lambda_i^T u_T^i,$$

where $\lambda_1^T, \ldots, \lambda_{m_T}^T \in \mathbb{K}$ and $u_T^1, \ldots, u_T^{m_T}$ are non degenerate surjections. We prove that, for every $1 \leq i \leq m_T$ and $2 \leq k \leq |T|$,

$$\pi_{\{k\}}(2 \cdot u_T^i) = 0.$$

It is true for $k = 2$, since $u_T^i \in \chi(2 < \cdots < |T|)$ so that there are at least two occurences of 2 in the surjection $2 \cdot u_T^i$. Suppose now that $k \geq 3$. We prove the statement by induction on $|T|$. If $|T| = 3$, the first tree of Example 2.3.2 gives $TR(\mu_T) = \pm 1232$ and

$\pi_{\{3\}}(232) = 22 = 0$. We now suppose that $|T| \geq 4$. By the proof of Lemma 2.3.5, we have

$$Tr(\mu_T) = -\sum_{S \subset T} \pm 12 \cdot \pi_2(TR(\mu_{T/S}) \circ_S TR(\mu_S)).$$

Let $S \subset T$ be such that $b_S = b_{T/S} = 1$. Suppose that $|S|, |T/S| \neq 2$. By Lemma 2.3.5, there exist $u_S \in \chi(V_S \setminus \{r(S)\})$ and $u_{T/S} \in \chi(V_{T/S} \setminus \{r(T/S)\})$ such that

$$\begin{array}{rcl} TR(\mu_S) & = & r(S)p \cdot u_S; \\ TR(u_{T/S}) & = & r(T/S)q \cdot u_{T/S} \end{array}$$

where $p \in V_S$ and $q \in V_{T/S}$ are the second element of their respective totally ordered set. If $r(S) \neq 1$, then $r(T/S) = 1$ and $q = 2$, since $b_T = 1$, so that

$$TR(\mu_{T/S}) \circ_S TR(\mu_S) = 12 \cdot (u_{T/S} \circ_S (r(S)p \cdot u_S))$$

whose associated term in the sum is 0. Suppose now that $r(S) = 1$. Then $r(T/S) = S$, so that

$$TR(\mu_{T/S}) \circ_S TR(\mu_S) = 1p \cdot u_S \cdot q \cdot u_{T/S}.$$

We write u_S and $u_{T/S}$ in the basis given by non degenerated surjections:

$$TR(\mu_{T/S}) \circ_S TR(\mu_S) = \sum_{i=1}^{m_S} \sum_{j=1}^{m_{T/S}} \lambda_i^S \lambda_j^{T/S} 1p \cdot u_S^i \cdot q \cdot u_{T/S}^j.$$

Since $k \neq 1, 2$, we have, for every $1 \leq i \leq m_S$ and $1 \leq j \leq m_{T/S}$,

$$\pi_{\{k\}}(12 \cdot \pi_2(1p \cdot u_S^i \cdot q \cdot u_{T/S}^j)) = 12 \cdot \pi_{\{k\}}(p \cdot u_S^i) \cdot \pi_{\{k\}}(q \cdot u_{T/S}^j).$$

By induction hypothesis (on S if $k \in V_S$, on T/S else), we obtain 0. Suppose now $|S| = 2$ and $|T/S| \neq 2$. By the same argument as before, we can restrict to the case $r(S) = 1$ (which implies that $r(T/S) = S$), so that

$$TR(\mu_{T/S}) \circ_S TR(\mu_S) = 1pq \cdot u_{T/S}.$$

where $p \in V_S$ and $q \in V_{T/S}$ are the second element of their respective totally ordered set. We have

$$12 \cdot \pi_2(TR(\mu_{T/S}) \circ_S TR(\mu_S)) = 12pq \cdot u_{T/S} = \sum_{i=1}^{m_{T/S}} \lambda_i^{T/S} 12pq \cdot u_{T/S}^i.$$

Let $1 \leq i \leq m_{T/S}$. If $p = 2$, then the corresponding term in the sum is 0. If $p \neq 2$, then $q = 2$ so that we need to compute

$$1 \cdot \pi_{\{k\}}(2p2 \cdot u_{T/S}^i).$$

If $k = p$, then $\pi_{\{p\}}(2p2 \cdot u_{T/S}^i) = 22 \cdot u_{T/S}^i = 0$. If $k \neq p$, then $\pi_{\{k\}}(2p2 \cdot u_{T/S}^i) = 2p2 \cdot \pi_{\{k\}}(u_{T/S}^i)$, which is 0 by induction hypothesis on T/S. Suppose now that $|T/S| = 2$ and $|S| \neq 2$. As before, we can suppose that $r(S) = 1$ and $2 \notin V_S$. We then have

$$TR(\mu_{T/S}) \circ_S TR(\mu_S) = 13 \cdot u_S \cdot 2,$$

which gives

$$12 \cdot \pi_2(TR(\mu_{T/S}) \circ_S TR(\mu_S)) = 123 \cdot u_S \cdot 2 = \sum_{i=1}^{m_S} \lambda_i^S 123 \cdot u_S^i \cdot 2.$$

Let $1 \leq i \leq m_S$. Then

$$\pi_{\{k\}}(23 \cdot u_S^i \cdot 2) = 2 \cdot \pi_{\{k\}}(3 \cdot u_S^i) \cdot 2 = 0,$$

by induction hypothesis on S. The case $|S| = |T/S| = 2$ gives $|T| = 3$ which has already be proved in the beginning of the proof.

We thus have proved that $\pi_{\{k\}}(2 \cdot u_T^i) = 0$ for every canonical tree T such that $|T| \geq 3$ and $2 \leq k \leq |T|, 1 \leq i \leq m_T$. We now prove the lemma. Let $2 \leq k \leq |T|$. By definition of the interval cut operations (see [BF04, §2.2.1]), the tensors with a factor of the form $\underline{0}, \dots, \underline{n}$ at position k occurring in the expansion of $(T \otimes \Lambda\mu_T)(\underline{0 \cdots n})$ are precisely produced by the surjections $12 \cdot u_T^1, \dots, 12 \cdot u_T^{m_T}$ which contain only one occurrence of k. Let $1 \leq i \leq m_T$ be such that u_T^i contains only one occurrence of k. The tensors produced by $2 \cdot u_T^i$ with a degree -1 element at position k are given by the insertion of the appropriate degree 0 vertex at position k of the tensors produced by the surjection $\pi_{\{k\}}(2 \cdot u_T^i)$. Since this surjection is 0, the lemma is proved. □

Lemma 2.4.12. *Let $n \geq 2$. Let $a, b_1, \dots, b_m \in A$, let $\underline{x}, \underline{y_1}, \dots, \underline{y_m} \in N_*(\Delta^n)$ be basis elements and $r_1, \dots, r_m \geq 0$. Suppose that*

$$|\underline{x}| + r_1|\underline{y_1}| + \cdots + r_m|\underline{y_m}| > n - 2.$$

Then $a \otimes \underline{x}^\vee \{\!\{ b_1 \otimes \underline{y_1}^\vee, \dots, b_m \otimes \underline{y_m}^\vee \}\!\}_{r_1,\dots,r_m} = 0$.

Proof. Let $r = r_1 + \cdots + r_m$. We more generally show that for every $\mu \in \Sigma^{-r}\mathcal{E}(r+1)_{r-1}$ and $\underline{z_1}, \dots, \underline{z_r} \in N_*(\Delta^n)$ such that $|\underline{x}| + |\underline{z_1}| + \cdots + |\underline{z_r}| > n - 2$, the evaluation of μ on the tensor $\underline{x}^\vee \otimes \underline{z_1}^\vee \otimes \cdots \otimes \underline{z_r}^\vee$ when using the $\mathcal{E}$-algebra structure of $N^*(\Delta^n)$ is 0. One one hand, the evaluation of μ on the tensor $\underline{x}^\vee \otimes \underline{z_1}^\vee \otimes \cdots \otimes \underline{z_r}^\vee$ is an element with degree $-1 - |\underline{x}| - |\underline{z_1}| - \cdots - |\underline{z_r}| < 1 - n$. On the other hand, since the result is an element of $\Sigma N^*(\Delta^n)$, its degree is equal or greater than $|\underline{0 \cdots n}^\vee| = 1 - n$. The evaluation of μ on the tensor $\underline{x}^\vee \otimes \underline{z_1}^\vee \otimes \cdots \otimes \underline{z_r}^\vee$ must then be 0.

To obtain the lemma, we apply this result to $\mu = \Lambda\mu_T$ where $T \in \mathcal{PRT}(r+1)$ is a canonical tree, and $\underline{z_1}, \dots, \underline{z_r} = \underbrace{\underline{y_1}, \dots, \underline{y_1}}_{r_1}, \dots, \underbrace{\underline{y_m}, \dots, \underline{y_m}}_{r_m}$ up to a shuffle permutation in $Sh(r_1, \dots, r_m)$. □

Before stating the next lemma, recall that if A is a brace algebra and $\tau \in \mathcal{MC}(A)$, then we have a differential defined by

$$d_\tau(x) = d(x) + \tau\langle x \rangle - (-1)^{|x|} x\langle \tau \rangle.$$

We denote by A^τ the underlying dg $\mathbb{K}$-module.

Lemma 2.4.13. *Let $\tau \in \mathcal{MC}(A)$ and $n \geq 1$. We denote by $\mathcal{MC}_{n+1}(A)_\tau$ the set given by elements $\xi \in \mathcal{MC}_{n+1}(A)$ with faces given by τ. Then we have a bijection*

$$f : Z_n(A^\tau) \longrightarrow \mathcal{MC}_{n+1}(A)_\tau$$

given by

$$f(h) = -\tau \otimes \left(\sum_{k=0}^{n+1} \underline{k}^\vee \right) - h \otimes \underline{0 \cdots (n+1)}^\vee.$$

Proof. Let $\xi \in \mathcal{MC}_{n+1}(A)_\tau$. Then there exists $h \in A_n$ such that

$$\xi = -\tau \otimes \left(\sum_{k=0}^{n+1} \underline{k}^\vee \right) - h \otimes \underline{0 \cdots (n+1)}^\vee.$$

We make precise the Maurer-Cartan condition on ξ. Let $p \geq 2$. By Lemma 2.4.11, we have

$$\begin{aligned}\xi\{\!\{\xi\}\!\}_p = \sum_{k=0}^{n+1} -(-1)^p \tau \otimes \underline{k}^\vee \{\!\{ h \otimes \underline{0 \cdots (n+1)}^\vee \}\!\}_p \\ + (-1)^{p+1} h \otimes \underline{0 \cdots (n+1)}^\vee \{\!\{ h \otimes \underline{0 \cdots (n+1)}^\vee \}\!\}_p.\end{aligned}$$

By Lemma 2.4.12, and since we have $n + np > -1 + np > n - 1$ because $p \geq 2$, we deduce that $\xi\{\!\{\xi\}\!\}_p = 0$. If $p = 1$, then, by Corollary 2.3.8,

$$\xi\{\!\{\xi\}\!\}_1 = -\tau\langle\tau\rangle \otimes \left(\sum_{k=0}^{n+1} \underline{k}^\vee \right) - (\tau\langle h \rangle - (-1)^n h\langle\tau\rangle) \otimes \underline{0 \cdots (n+1)}^\vee.$$

We also have

$$d(\xi) = -d(\tau) \otimes \left(\sum_{k=0}^{n+1} \underline{k}^\vee \right) - d(h) \otimes \underline{0 \cdots (n+1)}^\vee.$$

The Maurer-Cartan condition on ξ is then equivalent to

$$d(h) + \tau\langle h \rangle - (-1)^n h\langle\tau\rangle = 0.$$

which gives our desired bijection

$$f : Z_n(A^\tau) \longrightarrow \mathcal{MC}(A \otimes \Sigma N^*(\Delta^{n+1}))_\tau.$$

□

We now consider $n = 2$. The computation of π_2 will emphasize a group structure on $H_1(A^\tau)$ given by the following lemma.

Lemma 2.4.14. *Let A be a complete brace algebra and $\tau \in \mathcal{MC}(A)$. Then $(H_1(A^\tau), *_\tau, 0)$ is an abelian group with the product $*_\tau$ defined by*

$$[\mu] *_\tau [\mu'] = [\mu + \mu' + \tau\langle\mu, \mu'\rangle].$$

Proof. We first prove that if $\mu, \mu' \in Z_1(A^\tau)$ then $\mu'' := \mu + \mu' + \tau\langle\mu, \mu'\rangle \in Z_1(A^\tau)$. We have

$$\begin{aligned} d(\mu'') = d(\mu) + d(\mu') - \tau\langle\tau, \mu, \mu'\rangle + \tau\langle\mu, \tau, \mu'\rangle - \tau\langle\mu, \mu', \tau\rangle \\ - \tau\langle\tau\langle\mu\rangle, \mu'\rangle + \tau\langle\mu, \tau\langle\mu'\rangle\rangle - \tau\langle\tau\langle\mu, \mu'\rangle\rangle - \tau\langle d(\mu), \mu'\rangle + \tau\langle\mu, d(\mu')\rangle. \end{aligned}$$

We also have

$$\mu''\langle\tau\rangle = \mu\langle\tau\rangle + \mu'\langle\tau\rangle + \tau\langle\tau, \mu, \mu'\rangle - \tau\langle\mu, \tau, \mu'\rangle + \tau\langle\mu, \mu'\langle\tau\rangle\rangle - \tau\langle\mu\langle\tau\rangle, \mu'\rangle + \tau\langle\mu, \mu'\langle\tau\rangle\rangle$$

and

$$\tau\langle\mu''\rangle = \tau\langle\mu\rangle + \tau\langle\mu'\rangle + \tau\langle\tau\langle\mu, \mu'\rangle\rangle.$$

At the end, we obtain that $d_\tau(\mu'') = 0$, which proves that $\mu'' \in Z_1(A^\tau)$.

We now show that the product $*_\tau$ is well defined on $H_1(A^\tau)$. Let $\mu, \mu_1, \mu_2 \in Z_1(A^\tau)$ and $\psi \in A_2$ be such that

$$\mu_1 - \mu_2 = d(\psi) + \tau\langle\psi\rangle - \psi\langle\tau\rangle.$$

Let $\psi' := \psi + \tau\langle\mu, \psi\rangle$. We show that

$$\mu_1 - \mu_2 + \tau\langle\mu, \mu_1 - \mu_2\rangle = d(\psi') + \tau\langle\psi'\rangle - \psi'\langle\tau\rangle.$$

We first compute $d(\psi')$. We have

$$\begin{aligned} d(\psi') = d(\psi) - \tau\langle\tau, \mu, \psi\rangle + \tau\langle\mu, \tau, \psi\rangle + \tau\langle\mu, \psi, \tau\rangle - \tau\langle\tau\langle\mu\rangle, \psi\rangle + \tau\langle\mu, \tau\langle\psi\rangle\rangle \\ - \tau\langle\tau\langle\mu, \psi\rangle\rangle - \tau\langle d(\mu), \psi\rangle + \tau\langle\mu, d(\psi)\rangle. \end{aligned}$$

We also have

$$\psi'\langle\tau\rangle = \psi\langle\tau\rangle - \tau\langle\tau, \mu, \psi\rangle + \tau\langle\mu, \tau, \psi\rangle + \tau\langle\mu, \psi, \tau\rangle + \tau\langle\mu\langle\tau\rangle, \psi\rangle + \tau\langle\mu, \psi\langle\tau\rangle\rangle.$$

At the end, we obtain

$$d(\psi') + \tau\langle\psi'\rangle - \psi'\langle\tau\rangle = \mu_1 - \mu_2 + \tau\langle\mu, \mu_1 - \mu_2\rangle$$

so that

$$[\mu + \mu_1 + \tau\langle\mu, \mu_1\rangle] = [\mu + \mu_2 + \tau\langle\mu, \mu_2\rangle].$$

By the same computations with $\psi' := \psi - \tau\langle\psi, \mu\rangle$, we can show that

$$[\mu + \mu_1 + \tau\langle\mu_1, \mu\rangle] = [\mu + \mu_2 + \tau\langle\mu_2, \mu\rangle].$$

The product $*_\tau$ is thus well defined on $H_1(A^\tau)$. We now prove that it endows $H_1(A^\tau)$ with an abelian group structure. We prove the associativity of the operation $*_\tau$. We have

$$\begin{aligned} ([\mu] *_\tau [\mu']) *_\tau [\mu''] &= [\mu + \mu' + \mu'' + \tau\langle\mu, \mu'\rangle + \tau\langle\mu + \mu' + \tau\langle\mu, \mu'\rangle, \mu''\rangle]; \\ [\mu] *_\tau ([\mu'] *_\tau [\mu'']) &= [\mu + \mu' + \mu'' + \tau\langle\mu', \mu''\rangle + \tau\langle\mu, \mu' + \mu'' + \tau\langle\mu', \mu''\rangle\rangle]. \end{aligned}$$

The difference between the two representatives is

$$\tau\langle\mu, \tau\langle\mu', \mu''\rangle\rangle - \tau\langle\tau\langle\mu, \mu'\rangle, \mu''\rangle.$$

We show that this element is the image of $\psi := \tau\langle\mu, \mu', \mu''\rangle \in A_2$ under d_τ. First, using that $d(\tau) = -\tau\langle\tau\rangle$ and the brace algebra structure on A, we have

$$\begin{aligned} d(\tau)\langle\mu, \mu', \mu''\rangle = & -\tau\langle\tau, \mu, \mu', \mu''\rangle + \tau\langle\mu, \tau, \mu', \mu''\rangle - \tau\langle\mu, \mu', \tau, \mu''\rangle + \tau\langle\mu, \mu', \mu'', \tau\rangle \\ & - \tau\langle\tau\langle\mu\rangle, \mu', \mu''\rangle + \tau\langle\mu, \tau\langle\mu'\rangle, \mu''\rangle - \tau\langle\mu, \mu', \tau\langle\mu''\rangle\rangle \\ & - \tau\langle\tau\langle\mu, \mu'\rangle, \mu''\rangle + \tau\langle\mu, \tau\langle\mu', \mu''\rangle\rangle - \tau\langle\tau\langle\mu, \mu', \mu''\rangle\rangle. \end{aligned}$$

This gives

$$\begin{aligned} d(\psi) = & -\tau\langle\tau, \mu, \mu', \mu''\rangle + \tau\langle\mu, \tau, \mu', \mu''\rangle - \tau\langle\mu, \mu', \tau, \mu''\rangle + \tau\langle\mu, \mu', \mu'', \tau\rangle \\ & - \tau\langle\tau\langle\mu\rangle, \mu', \mu''\rangle + \tau\langle\mu, \tau\langle\mu'\rangle, \mu''\rangle - \tau\langle\mu, \mu', \tau\langle\mu''\rangle\rangle \\ & - \tau\langle\tau\langle\mu, \mu'\rangle, \mu''\rangle + \tau\langle\mu, \tau\langle\mu', \mu''\rangle\rangle - \tau\langle\tau\langle\mu, \mu', \mu''\rangle\rangle \\ & - \tau\langle d(\mu), \mu', \mu''\rangle + \tau\langle\mu, d(\mu'), \mu''\rangle - \tau\langle\mu, \mu', d(\mu'')\rangle \end{aligned}$$

Using that $\mu, \mu', \mu'' \in Z_1(A^\tau)$, we obtain

$$\begin{aligned} d(\psi) = & -\tau\langle\tau, \mu, \mu', \mu''\rangle + \tau\langle\mu, \tau, \mu', \mu''\rangle - \tau\langle\mu, \mu', \tau, \mu''\rangle + \tau\langle\mu, \mu', \mu'', \tau\rangle \\ & + \tau\langle\mu\langle\tau\rangle, \mu', \mu''\rangle - \tau\langle\mu, \mu'\langle\tau\rangle, \mu''\rangle + \tau\langle\mu, \mu', \mu''\langle\tau\rangle\rangle \\ & - \tau\langle\tau\langle\mu, \mu'\rangle, \mu''\rangle + \tau\langle\mu, \tau\langle\mu', \mu''\rangle\rangle - \tau\langle\tau\langle\mu, \mu', \mu''\rangle\rangle. \end{aligned}$$

We also have

$$\begin{aligned} \psi\langle\tau\rangle = & -\tau\langle\tau, \mu, \mu', \mu''\rangle + \tau\langle\mu, \tau, \mu', \mu''\rangle - \tau\langle\mu, \mu', \tau, \mu''\rangle + \tau\langle\mu, \mu', \mu'', \tau\rangle \\ & + \tau\langle\mu\langle\tau\rangle, \mu', \mu''\rangle - \tau\langle\mu, \mu'\langle\tau\rangle, \mu''\rangle + \tau\langle\mu, \mu', \mu''\langle\tau\rangle\rangle. \end{aligned}$$

which finally gives

$$d(\psi) + \tau\langle\psi\rangle - \psi\langle\tau\rangle = \tau\langle\mu, \tau\langle\mu', \mu''\rangle\rangle - \tau\langle\tau\langle\mu, \mu'\rangle, \mu''\rangle$$

so that we have the associativity.

We now prove that every element $[\mu]$ has an inverse under $*_\tau$. We set $\mu'_0 = -\mu$ and, for every $n \geq 0$,

$$\mu'_{n+1} = -\mu - \tau\langle\mu, \mu'_n\rangle.$$

We obtain a Cauchy sequence in A. Because A is complete, this sequence has a limit denoted by μ' which satisfies

$$\mu + \mu' + \tau\langle\mu, \mu'\rangle = 0$$

so that $[\mu']$ is the inverse of $[\mu]$ under $*_\tau$. We thus have proved that $*_\tau$ endows $H_1(A^\tau)$ with a group structure.

We now prove that $*_\tau$ is abelian. Let $\mu, \mu' \in A_1$. We set $\psi := \mu\langle\mu'\rangle$, and prove that

$$d(\psi) + \tau\langle\psi\rangle - \psi\langle\tau\rangle = \tau\langle\mu', \mu\rangle - \tau\langle\mu, \mu'\rangle.$$

We have

$$\begin{array}{rcl} d(\psi) & = & d(\mu)\langle\mu'\rangle - \mu\langle d(\mu')\rangle \\ & = & -\tau\langle\mu\rangle\langle\mu'\rangle - \mu\langle\tau\rangle\langle\mu'\rangle + \mu\langle\tau\langle\mu'\rangle\rangle + \mu\langle\mu'\langle\tau\rangle\rangle \\ & = & -\tau\langle\mu\langle\mu'\rangle\rangle - \tau\langle\mu, \mu'\rangle + \tau\langle\mu', \mu\rangle - \mu\langle\tau\langle\mu'\rangle\rangle \\ & & -\mu\langle\tau, \mu'\rangle + \mu\langle\mu', \tau\rangle + \mu\langle\tau\langle\mu'\rangle\rangle + \mu\langle\mu'\langle\tau\rangle\rangle \\ & = & -\tau\langle\mu\langle\mu'\rangle\rangle - \tau\langle\mu, \mu'\rangle + \tau\langle\mu', \mu\rangle \\ & & -\mu\langle\tau, \mu'\rangle + \mu\langle\mu', \tau\rangle + \mu\langle\mu'\langle\tau\rangle\rangle \end{array}$$

and

$$\psi\langle\tau\rangle = \mu\langle\mu', \tau\rangle - \mu\langle\tau, \mu'\rangle + \mu\langle\mu'\langle\tau\rangle\rangle.$$

We then have

$$d(\psi) + \tau\langle\psi\rangle - \psi\langle\tau\rangle = \tau\langle\mu', \mu\rangle - \tau\langle\mu, \mu'\rangle$$

which proves that

$$[\mu + \mu' + \tau\langle\mu, \mu'\rangle] = [\mu' + \mu + \tau\langle\mu', \mu\rangle].$$

The operation $*_\tau$ is then commutative.

The lemma is proved. □

Theorem 2.4.15. *Let A be a complete brace algebra and $\tau \in \mathcal{MC}(A)$. Then*

$$\pi_2(\mathcal{MC}_\bullet(A), \tau) \simeq (H_1(A^\tau), *_\tau, 0).$$

Proof. By Lemma 2.4.13, we have a bijection

$$f : Z_1(A^\tau) \longrightarrow \mathcal{MC}_2(A)_\tau.$$

We consider its composite $\widetilde{f} : Z_1(A^\tau) \longrightarrow \pi_2(\mathcal{MC}_\bullet(A), \tau)$ with the projection of $\mathcal{MC}_2(A)_\tau$ onto $\pi_2(\mathcal{MC}_\bullet(A), \tau)$. We show that $\widetilde{f}$ is compatible with the equivalence relation on $H_1(A^\tau)$ given by Lemma 2.4.14. Let $\mu, \mu' \in Z_1(A^\tau)$ be such that there exists $\psi \in A_2$ with $\mu - \mu' = d_\tau(\psi)$. Namely,

$$d(\psi) + \tau\langle\psi\rangle - \psi\langle\tau\rangle = \mu - \mu'.$$

By Corollary 2.3.8, Corollary 2.3.9 and Lemma 2.4.12, we have

$$-\tau \otimes (\underline{0}^\vee + \underline{1}^\vee + \underline{2}^\vee) - \mu \otimes \underline{123}^\vee - \mu' \otimes \underline{023}^\vee + \psi \otimes \underline{0123}^\vee \in \mathcal{MC}(A \otimes \Sigma N^*(\Delta^3)),$$

which shows that $\widetilde{f}(\mu) = \widetilde{f}(\mu')$. We thus have a well defined map

$$\overline{f} : H_1(A^\tau) \longrightarrow \pi_2(\mathcal{MC}_\bullet(A), \tau).$$

We prove that $\overline{f}$ preserves the group structures. Let $\mu, \mu' \in Z_1(A^\tau)$. Recall that

$$f(\mu) = -\tau \otimes (\underline{0}^\vee + \underline{1}^\vee + \underline{2}^\vee) - \mu \otimes \underline{012}^\vee;$$

$$f(\mu') = -\tau \otimes (\underline{0}^\vee + \underline{1}^\vee + \underline{2}^\vee) - \mu' \otimes \underline{012}^\vee.$$

We search for $\mu'' \in A_1$ and $\psi \in A_2$ such that

$$\begin{aligned}\omega := -\tau \otimes (\underline{0}^\vee + \underline{1}^\vee + \underline{2}^\vee + \underline{3}^\vee) - \mu' \otimes \underline{123}^\vee \\ - \mu'' \otimes \underline{023}^\vee - \mu \otimes \underline{013}^\vee + \psi \otimes \underline{0123}^\vee \in \mathcal{MC}(A \otimes \Sigma N^*(\Delta^3)).\end{aligned}$$

We have

$$d(\omega) = -d(\tau) \otimes (\underline{0}^\vee + \underline{1}^\vee + \underline{2}^\vee + \underline{3}^\vee) - d(\mu') \otimes \underline{123}^\vee - d(\mu'') \otimes \underline{023}^\vee - d(\mu) \otimes \underline{013}^\vee$$

$$+(d(\psi) - \mu'' + \mu + \mu') \otimes \underline{0123}^\vee.$$

By Corollary 2.3.8, we also have

$$\omega\{\!\{\omega\}\!\}_1 = -\tau\langle\tau\rangle \otimes (\underline{0}^\vee + \underline{1}^\vee + \underline{2}^\vee + \underline{3}^\vee) - (\tau\langle\mu'\rangle + \mu'\langle\tau\rangle) \otimes \underline{123}^\vee - (\tau\langle\mu''\rangle + \mu''\langle\tau\rangle) \otimes \underline{023}^\vee$$

$$-(\tau\langle\mu\rangle + \mu\langle\tau\rangle) \otimes \underline{013}^\vee + (\tau\langle\psi\rangle - \psi\langle\tau\rangle) \otimes \underline{0123}^\vee.$$

By Corollary 2.3.9, we have

$$\omega\{\!\{\omega\}\!\}_2 = \tau\langle\mu, \mu'\rangle \otimes \underline{0123}^\vee.$$

Finally, for every $p > 2$ and by Lemma 2.4.11,

$$\begin{aligned}\omega\{\!\{\omega\}\!\}_p &= -\tau \otimes (\underline{0}^\vee + \underline{1}^\vee + \underline{2}^\vee + \underline{3}^\vee)\{\!\{-\mu' \otimes \underline{123}^\vee - \mu'' \otimes \underline{023}^\vee - \mu \otimes \underline{013}^\vee + \psi \otimes \underline{0123}^\vee\}\!\}_p \\ &= \sum_{s+t=p} \tau \otimes (\underline{0}^\vee + \underline{1}^\vee + \underline{2}^\vee + \underline{3}^\vee)\{\!\{-\mu' \otimes \underline{123}^\vee \\ &\qquad -\mu'' \otimes \underline{023}^\vee - \mu \otimes \underline{013}^\vee, \psi \otimes \underline{0123}^\vee\}\!\}_{s,t}.\end{aligned}$$

Since $p > 2$, for every $s, t \geq 0$ such that $s + t = p$, we have $2s + 3t > p + 2$. From Lemma 2.4.12, we deduce $\omega\{\!\{\omega\}\!\}_p = 0$. We then see that ω is a Maurer-Cartan element if and only if

$$d_\tau(\psi) - \mu'' + \mu + \mu' + \tau\langle\mu, \mu'\rangle = 0.$$

If we set $\psi = 0$ and $\mu'' = \mu + \mu' + \tau\langle\mu, \mu'\rangle$, this shows that

$$[f(\mu)] \cdot [f(\mu')] = [-\tau \otimes (\underline{0}^\vee + \underline{1}^\vee + \underline{2}^\vee) - (\mu + \mu' + \tau\langle\mu, \mu'\rangle) \otimes \underline{012}^\vee]$$

in $\pi_2(\mathcal{MC}_\bullet(A), \tau)$. We thus have proved

$$\overline{f}([\mu]) \cdot \overline{f}([\mu']) = \overline{f}([\mu] *_\tau [\mu']).$$

The morphism $\overline{f}$ is surjective, since $\widetilde{f}$ is bijective. It is also injective. Indeed, the equation $\overline{f}([\mu]) = 0$ is equivalent to $[\mu] = 0$, according to the beginning of the proof of this theorem with $\mu' = 0$. The map $\overline{f}$ is thus an isomorphism, which proves the theorem. □

Computation of $\pi_n(\mathcal{MC}_\bullet(A), \tau)$ for $n \geq 3$

We finally compute the groups $\pi_n(\mathcal{MC}_\bullet(A), \tau)$ for every $n \geq 3$.

Theorem 2.4.16. *Let A be a complete brace algebra and $\tau \in \mathcal{MC}(A)$. Then, for all $n \geq 3$, we have an isomorphism of groups*

$$\pi_{n+1}(\mathcal{MC}_\bullet(A), \tau) \simeq H_n(A^\tau).$$

Proof. By Lemma 2.4.13, we have a bijection $f : Z_n(A^\tau) \longrightarrow \mathcal{MC}_{n+1}(A)_\tau$. Consider its composite $\widetilde{f} : Z_n(A^\tau) \longrightarrow \pi_{n+1}(\mathcal{MC}_\bullet(A), \tau)$ with the projection of $\mathcal{MC}_{n+1}(A)_\tau$ onto $\pi_{n+1}(\mathcal{MC}_\bullet(A), \tau)$. We show that $\widetilde{f}$ is a morphism of groups. Let $\mu, \mu' \in Z_n(A^\tau)$. We set

$$\omega = -\tau \otimes \left(\sum_{k=0}^{n} \underline{k}^\vee\right) - \mu \otimes \underline{0 \cdots (n+1)}^\vee;$$

$$\omega' = -\tau \otimes \left(\sum_{k=0}^{n} \underline{k}^\vee\right) - \mu' \otimes \underline{0 \cdots (n+1)}^\vee.$$

We compute $[\omega] + [\omega']$ in $\pi_{n+1}(\mathcal{MC}_\bullet(A), \tau)$. This is equivalent to searching $\mu'' \in Z_n(A^\tau)$ and $\psi \in A_{n+1}$ such that

$$\begin{aligned}\xi := -\tau \otimes \left(\sum_{k=0}^{n+2} \underline{k}^\vee\right) - \mu \otimes \underline{1 \cdots (n+2)}^\vee - \mu'' \otimes \underline{02 \cdots (n+2)}^\vee \\ - \mu' \otimes \underline{013 \cdots (n+2)}^\vee + \psi \otimes \underline{0 \cdots (n+2)}^\vee \in \mathcal{MC}(A \otimes \Sigma N^*(\Delta^{n+2})).\end{aligned}$$

We make precise the Maurer-Cartan condition on ξ. We first compute $d(\xi)$. Note that we have, for every $0 \leq k \leq n+2$,

$$d(\underline{k}^\vee) = \sum_{k<j\leq n+2} \underline{kj}^\vee - \sum_{0\leq i<k} \underline{ik}^\vee,$$

which implies

$$\sum_{k=0}^{n+2} d(\underline{k}^\vee) = 0$$

by a variable substitution. We then have

$$\begin{aligned}d(\xi) = -d(\tau) \otimes \left(\sum_{k=0}^{n+2} \underline{k}^\vee\right) - d(\mu) \otimes \underline{1 \cdots (n+2)}^\vee - d(\mu'') \otimes \underline{02 \cdots (n+2)}^\vee \\ - d(\mu') \otimes \underline{013 \cdots (n+2)}^\vee + (d(\psi) + \mu'' - \mu - \mu') \otimes \underline{0 \cdots (n+2)}^\vee.\end{aligned}$$

We now compute $\xi\{\!\{\xi\}\!\}_1$. By Corollary 2.3.8, we have

$$\begin{aligned}\xi\{\!\{\xi\}\!\}_1 = -\tau\langle\tau\rangle \otimes \left(\sum_{k=0}^{n+2} \underline{k}^\vee\right) - (\tau\langle\mu\rangle - (-1)^n \mu\langle\tau\rangle) \otimes \underline{1 \cdots (n+2)}^\vee \\ - (\tau\langle\mu''\rangle - (-1)^n \mu''\langle\tau\rangle) \otimes \underline{02 \cdots (n+2)}^\vee \\ - (\tau\langle\mu'\rangle - (-1)^n \mu'\langle\tau\rangle) \otimes \underline{013 \cdots (n+2)}^\vee \\ + (\tau\langle\psi\rangle - (-1)^{n+1} \psi\langle\tau\rangle) \otimes \underline{0 \cdots (n+2)}^\vee.\end{aligned}$$

We now show that $\xi\{\!\{\xi\}\!\}_p = 0$ for every $p \geq 2$. By Lemma 2.4.11, we have

$$
\begin{aligned}
\xi\{\!\{\xi\}\!\}_p = &- \sum_{s+t=p} \tau \otimes \left(\sum_{k=0}^{n+2} \underline{k}^\vee\right) \{\!\{\mu \otimes \underline{1\cdots(n+2)}^\vee - \mu'' \otimes \underline{02\cdots(n+2)}^\vee \\
&\qquad - \mu' \otimes \underline{013\cdots(n+2)}^\vee, \psi \otimes \underline{0\cdots(n+2)}^\vee\}\!\}_{s,t} \\
&+ \sum_{s+t=p} (\mu\otimes\underline{1\cdots(n+2)}^\vee - \mu''\otimes\underline{02\cdots(n+2)}^\vee - \mu'\otimes\underline{013\cdots(n+2)}^\vee)\{\!\{\mu\otimes\underline{1\cdots(n+2)}^\vee \\
&\qquad - \mu'' \otimes \underline{02\cdots(n+2)}^\vee - \mu' \otimes \underline{013\cdots(n+2)}^\vee, \psi \otimes \underline{0\cdots(n+2)}^\vee\}\!\}_{s,t} \\
&\qquad + \sum_{s+t=p} \psi \otimes \underline{0\cdots(n+2)}^\vee \{\!\{\mu \otimes \underline{1\cdots(n+2)}^\vee - \mu'' \otimes \underline{02\cdots(n+2)}^\vee \\
&\qquad\qquad - \mu' \otimes \underline{013\cdots(n+2)}^\vee, \psi \otimes \underline{0\cdots(n+2)}^\vee\}\!\}_{s,t}
\end{aligned}
$$

Since $n \geq 3$ and $p \geq 2$, we can apply Lemma 2.4.12 to obtain $\xi\{\!\{\xi\}\!\}_p = 0$. At the end, since $\mu, \mu', h \in Z_n(A^\tau)$, we have

$$
d(\xi) + \sum_{p\geq 1} \xi\{\!\{\xi\}\!\}_p = (d(\psi) + \tau\langle\psi\rangle - (-1)^{n+1}\psi\langle\tau\rangle - \mu - \mu' + \mu'') \otimes \underline{0\cdots(n+2)}^\vee.
$$

If we set $\mu'' := \mu + \mu' \in Z_n(A^\tau)$ and $\psi := 0$, we then obtain that $\xi \in \mathcal{MC}(A \otimes \Sigma N^*(\Delta^{n+2}))$. We thus have proved that

$$
[\omega] + [\omega'] = \left[-\tau \otimes \left(\sum_{k=1}^{n+1} \underline{k}^\vee\right) - (\mu + \mu') \otimes \underline{0\cdots(n+1)}^\vee\right],
$$

which gives $\widetilde{f}(\mu + \mu') = \widetilde{f}(\mu) + \widetilde{f}(\mu')$. Now, because f is a bijection, we only need to prove that the kernel of $\widetilde{f}$ is exactly given by $d_\tau(A_{n+1})$. Let $\mu \in Z_n(A^\tau)$ and $\psi \in A_{n+1}$. By the previous computations, we see that the equation

$$
d(\psi) + \tau\langle\psi\rangle - (-1)^{n+1}\psi\langle\tau\rangle = \mu
$$

is equivalent to the assumption

$$
-\tau \otimes \left(\sum_{k=0}^{n+2} \underline{k}^\vee\right) - \mu \otimes \underline{1\cdots(n+2)}^\vee + \psi \otimes \underline{0\cdots(n+2)}^\vee \in \mathcal{MC}(A \otimes \Sigma N^*(\Delta^{n+2})),
$$

which shows that $\widetilde{f}(\mu) = 0$ if and only if $\mu = d_\tau(\psi)$ for some $\psi \in A_{n+1}$. We thus have an isomorphism

$$
\overline{f} : H_n(A^\tau) \xrightarrow{\simeq} \pi_{n+1}(\mathcal{MC}_\bullet(A), \tau).
$$

□

2.4.3 Remarks: interpretation of the low dimensional twisting coderivations

In this subsection, we give an interpretation of the differentials $\partial^0, \partial^1, \partial^2$ and ∂^3 computed in Lemma 2.3.6 and Corollary 2.3.9. This interpretation will be obtained by the study of the first simplices associated to the Maurer-Cartan simplicial set of $\mathrm{Hom}(\Lambda^{-1}\mathcal{A}s^\vee, \mathrm{End}_A)$ for some $A \in \mathrm{dgMod}_{\mathbb{K}}$.

Recall that for every non-symmetric cooperad $\mathcal{C}$ and non-symmetric operad $\mathcal{P}$ such that $\mathcal{C}(0) = \mathcal{P}(0) = 0$, the sequence $\mathrm{Hom}(\mathcal{C}, \mathcal{P})$ is endowed with the structure of an operad such that, for every $f \in \mathrm{Hom}(\mathcal{C}, \mathcal{P})(k), g_1 \in \mathrm{Hom}(\mathcal{C}, \mathcal{P})(i_1), \ldots, g_k \in \mathrm{Hom}(\mathcal{C}, \mathcal{P})(i_k)$ with $n = i_1 + \cdots + i_k$, the composition $\gamma(f \otimes g_1 \otimes \cdots \otimes g_k)$ is given by the composite

$$\begin{tikzcd}
\mathcal{C}(n) \arrow[r, "\Delta"] & \mathcal{C} \circ \mathcal{C}(n) \arrow[r, two heads] & \mathcal{C}(k) \otimes \mathcal{C}(i_1) \otimes \cdots \otimes \mathcal{C}(i_k) \arrow[d, "f \otimes g_1 \otimes \cdots \otimes g_k"] & & \\
& & \mathcal{P}(k) \otimes \mathcal{P}(i_1) \otimes \cdots \otimes \mathcal{P}(i_k) \arrow[r, hook] & \mathcal{P} \circ \mathcal{P}(n) \arrow[r, "\gamma"] & \mathcal{P}(n)
\end{tikzcd}.$$

From [GV95, Proposition 1], we deduce that $\bigoplus_{n \geq 1} \mathrm{Hom}(\mathcal{C}(n), \mathcal{P}(n))$ is endowed with the structure of a brace algebra. The braces are given by

$$f\langle g_1, \ldots, g_n \rangle = \sum_{1 \leq i_1 < \cdots < i_n \leq r} \gamma(f \otimes id \otimes \cdots \otimes \underset{i_1}{g_1} \otimes \cdots \otimes \underset{i_n}{g_n} \otimes \cdots \otimes id)$$

where $f \in \mathrm{Hom}(\mathcal{C}(r), \mathcal{P}(r)), g_1 \in \mathrm{Hom}(\mathcal{C}(m_1), \mathcal{P}(m_1)), \ldots, g_n \in \mathrm{Hom}(\mathcal{C}(m_n), \mathcal{P}(m_n))$. We immediatly see that $\bigoplus_{n \geq 2} \mathrm{Hom}(\mathcal{C}(n), \mathcal{P}(n))$ is a sub brace algebra of $\bigoplus_{n \geq 1} \mathrm{Hom}(\mathcal{C}(n), \mathcal{P}(n))$. Since $\prod_{n \geq 2} \mathrm{Hom}(\mathcal{C}(n), \mathcal{P}(n))$ is the completion of $\bigoplus_{n \geq 2} \mathrm{Hom}(\mathcal{C}(n), \mathcal{P}(n))$ under the filtration defined by

$$F_p(\bigoplus_{n \geq 2} \mathrm{Hom}(\mathcal{C}(n), \mathcal{P}(n))) := \bigoplus_{n \geq p+1} \mathrm{Hom}(\mathcal{C}(n), \mathcal{P}(n)),$$

we have that the above brace algebra structure on $\bigoplus_{n \geq 1} \mathrm{Hom}(\mathcal{C}(n), \mathcal{P}(n))$ induces a complete brace algebra structure on $\prod_{n \geq 2} \mathrm{Hom}(\mathcal{C}(n), \mathcal{P}(n))$.

We now consider the non-symmetric operad $\mathcal{A}s$ such that $\mathcal{A}s(0) = 0$ and $\mathcal{A}s(n) = \mathbb{K}$ for every $n \geq 1$ with trivial operadic compositions. Since $\mathcal{A}s$ is self-dual for Koszul duality (see for instance [LV12, Proposition 9.1.9]), the operad $\mathcal{A}s_\infty = B^c(\Lambda^{-1}\mathcal{A}s^\vee)$ encodes associative algebras up to homotopy. We apply the above analysis with $\mathcal{C} = \Lambda^{-1}\mathcal{A}s^\vee$ and $\mathcal{P} = \mathrm{End}_A$ for some $A \in \mathrm{dgMod}_{\mathbb{K}}$ in order to study morphisms from $\mathcal{A}s_\infty$ to End_A, or equivalently associative up to homotopy algebra structures on A. Note that we have an isomorphism of operads

$$\mathrm{Hom}(\Lambda^{-1}\mathcal{A}s^\vee, \mathrm{End}_A) \simeq \mathrm{End}_{\Sigma A}.$$

We set

$$\overline{B}(A) = \bigoplus_{n \geq 1} (\Sigma A)^{\otimes n} \ ; \ B_{\geq 2}(A) = \bigoplus_{n \geq 2} (\Sigma A)^{\otimes n}$$

so that $\overline{B}(A) = \Sigma A \oplus B_{\geq 2}(A)$. Let d be the differential of $\overline{B}(A)$ obtained from the internal differential of A by the Leibniz rule. Recall that $\overline{B}(A)$ is a coalgebra with as coproduct

$$\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{k=1}^{n-1} (a_1 \otimes \cdots \otimes a_k) \otimes (a_{k+1} \otimes \cdots \otimes a_n)$$

for every $n \geq 2$ and $a_1, \ldots, a_n \in \Sigma A$. The above isomorphism of operads provides a complete brace algebra structure on $\mathrm{Hom}(B_{\geq 2}(A), \Sigma A) \simeq \prod_{n\geq 2} \mathrm{End}_{\Sigma A}(n)$. Note that we have the isomorphism $\mathrm{Hom}(\overline{B}(A), \Sigma A) \simeq \mathrm{Hom}(\Sigma A, \Sigma A) \oplus \mathrm{Hom}(B_{\geq 2}(A), \Sigma A)$. In the following, we denote by $1 \in \mathrm{Hom}(\Sigma A, \Sigma A)$ the identity morphism so that we have a natural inclusion $\mathbb{K}1 \oplus \mathrm{Hom}(B_{\geq 2}(A), \Sigma A) \subset \mathrm{Hom}(\overline{B}(A), \Sigma A)$.

Proposition 2.4.17. *Giving a Maurer-Cartan element $\phi \in \mathcal{MC}(\mathrm{Hom}(B_{\geq 2}(A), \Sigma A))$ is equivalent to giving a coderivation of coalgebra of the form $d + \partial_\phi$ on $\overline{B}(A)$, where ∂_ϕ is the morphism obtained from ϕ by the Leibniz rule in the coalgebra $\overline{B}(A)$.*

Proof. Let $\phi \in \mathrm{Hom}(B_{\geq 2}(A), \Sigma A)$ be a degree -1 morphism. Then $(d + \partial_\phi)^2 = 0$ if and only if $d(\phi) + \phi\partial_\phi = 0$. By definition of ∂_ϕ, we have, for every $a_1, \ldots, a_n \in \Sigma A$,

$$\partial_\phi(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n} \sum_{j=1}^{n-i} \pm a_1 \otimes \cdots \otimes a_{i-1} \otimes \phi(a_i \otimes \cdots \otimes a_{i+j}) \otimes a_{i+j+1} \otimes \cdots \otimes a_n,$$

which gives $\phi\partial_\phi = \phi\langle\phi\rangle$. We thus have obtained that $d + \partial_\phi$ is a derivation of coalgebra if and only if $\phi \in \mathcal{MC}(\mathrm{Hom}(\overline{B}(A), \Sigma A))$. □

Since giving a morphism of operads $\mathcal{A}s_\infty \longrightarrow \mathrm{End}_A$ is equivalent to giving a Maurer-Cartan element in $\mathrm{Hom}(B_{\geq 2}(A), \Sigma A)$, we have the following classical definition.

Definition 2.4.18. *An* associative algebra up to homotopy *is a pair (A, ϕ) where A is a dg $\mathbb{K}$-module and $\phi \in \mathcal{MC}(\mathrm{Hom}(B_{\geq 2}(A), \Sigma A))$.*

For every $\phi \in \mathcal{MC}(\mathrm{Hom}(B_{\geq 2}(A), \Sigma A))$, we denote by $\overline{B}(A, \phi)$ the dg $\mathbb{K}$-module $\overline{B}(A)$ endowed with the coderivation $d + \partial_\phi$.

Definition 2.4.19. *Let (A_1, ϕ_1) and (A_2, ϕ_2) be two associative algebras up to homotopy. An ∞-morphism $f : (A_1, \phi_1) \to (A_2, \phi_2)$ is a morphism of coalgebras $f : \overline{B}(A, \phi_1) \longrightarrow \overline{B}(A, \phi_2)$ which commutes with the coderivations.*

In the following, we consider the category of associative algebras up to homotopy with set morphisms the ∞-morphisms.

Remark 2.4.20. *Note that since $B(A_2)$ is cofree, giving a morphism of coalgebras $\overline{B}(A_1) \longrightarrow \overline{B}(A_2)$ is equivalent to giving a morphism $\overline{B}(A_1) \longrightarrow \Sigma A_2$.*

Proposition 2.4.21. *Let $\phi_0, \phi_1 \in \mathcal{MC}(\mathrm{Hom}(\overline{\overline{B}}(A), \Sigma A))$. Then giving*

$$-\phi_0 \otimes \underline{0}^\vee - \phi_1 \otimes \underline{1}^\vee - \phi_{01} \otimes \underline{01}^\vee \in \mathcal{MC}_1(\mathrm{Hom}(B_{\geq 2}(A), \Sigma A))$$

is equivalent to giving a morphism of coalgebras

$$\Phi_{01} : \overline{B}(A, \phi_1) \longrightarrow \overline{B}(A, \phi_0)$$

which is the identity on $\Sigma A \subset \overline{B}(A)$.

Proof. Let $\omega := -\phi_0 \otimes \underline{0}^\vee - \phi_1 \otimes \underline{1}^\vee - \phi_{01} \otimes \underline{01}^\vee \in \mathrm{Hom}(B_{\geq 2}(A), \Sigma A) \otimes \Sigma N^*(\Delta^1)$. Let $\Phi_{01} : \overline{B}(A) \longrightarrow \overline{B}(A)$ be the unique morphism of coalgebras such that its composite with the projection $\pi_{\Sigma A} : \overline{B}(A) \longrightarrow \Sigma A$ is $1 + \phi_{01}$. We characterize the equation

$$(d + \partial_{\phi_0})\Phi_{01} = \Phi_{01}(d + \partial_{\phi_1}).$$

Since $\overline{B}(A)$ is cofree, this identity is equivalent to

$$\pi_{\Sigma A}(d + \partial_{\phi_0})\Phi_{01} = \pi_{\Sigma A}\Phi_{01}(d + \partial_{\phi_1}),$$

and then to

$$d(\phi_{01}) = \phi_1 + \phi_{01}\partial_{\phi_1} - \phi_0\Phi_{01}.$$

We precisely have $\phi_{01}\partial_{\phi_1} = \phi_{01}\langle\phi_1\rangle$ and $\phi_0\Phi_{01} = \phi_0 \circledcirc (1 + \phi_{01})$ by definition of ∂_{ϕ_1} and Φ_{01}. We thus have obtained that Φ_{01} commutes with the differentials if and only if

$$d(\phi_{01}) = \phi_1 + \phi_{01}\langle\phi_1\rangle - \phi_0 \circledcirc (1 + \phi_{01}).$$

By Lemma 2.4.7, this identity is equivalent to ask $\omega \in \mathcal{MC}_1(\mathrm{Hom}(B_{\geq 2}(A), \Sigma A))$, which proves the proposition. $\square$

We now characterize elements of $\mathcal{MC}_2(\mathrm{Hom}(B_{\geq 2}(A), \Sigma A))$. First, note that for every associative algebra E, every $\phi \in \mathcal{MC}(\mathrm{Hom}(B_{\geq 2}(A), \Sigma A))$ induces an element in $\mathcal{MC}(\mathrm{Hom}(B_{\geq 2}(A \otimes E), \Sigma(A \otimes E)))$, which we still denote by ϕ, and which is defined by applying ϕ on the left, and the algebra structure of E on the right. In particular, for every morphism of associative algebras $f : E \longrightarrow E'$, we have an ∞-morphism $id \otimes f : (A \otimes E, \phi) \longrightarrow (A \otimes E', \phi)$.

Next, recall that, for every $n \geq 0$, the dg $\mathbb{K}$-module $N^*(\Delta^n)$ is endowed with the structure of an associative algebra. This associative algebra structure is obtained by the dualization of the coassociative coalgebra structure on $N_*(\Delta^n)$ given by the Alexander-Whitney diagonal $AW : N_*(\Delta^n) \longrightarrow N_*(\Delta^n) \otimes N_*(\Delta^n)$ which is the operation given by the permutation $(12) \in \mathcal{E}(2)_0$. Explicitly, we have

$$AW(\underline{a_0 \cdots a_d}) = \sum_{k=0}^{d} \underline{a_0 \cdots a_k} \otimes \underline{a_k \cdots a_d},$$

for every $0 \leq a_0 < \cdots < a_d \leq n$.

Proposition 2.4.22. *Giving a Maurer-Cartan element in* $\mathcal{MC}_2(\mathrm{Hom}(B_{\geq 2}(A), \Sigma A))$ *is equivalent to giving Maurer-Cartan elements* $\phi_0, \phi_1, \phi_2 \in \mathcal{MC}(\mathrm{Hom}(B_{\geq 2}(A), \Sigma A))$ *and a diagram of the form*

$$\begin{tikzcd} & (A,\phi_1) \arrow[dr, "\Phi_{01}"] \arrow[d, phantom, "\Downarrow \Phi_{012}"] & \\ (A,\phi_2) \arrow[ur, "\Phi_{12}"] \arrow[rr, "\Phi_{02}"'] & {} & (A,\phi_0) \end{tikzcd}$$

in the category of A_∞*-algebras, where* $\Phi_{012} : (A, \phi_2) \longrightarrow (A \otimes N^*(\Delta^1), \phi_0)$ *is a homotopy from* $\Phi_{01}\Phi_{12}$ *to* Φ_{02}.

Proof. We consider

$$\omega := -\phi_0 \otimes \underline{0}^\vee - \phi_1 \otimes \underline{1}^\vee - \phi_2 \otimes \underline{2}^\vee - \phi_{01} \otimes \underline{01}^\vee - \phi_{02} \otimes \underline{02}^\vee - \phi_{12} \otimes \underline{02}^\vee - \phi_{012} \otimes \underline{012}^\vee.$$

We characterize the Maurer-Cartan condition on ω. By definition of the $\Gamma\Lambda\mathcal{PL}_\infty$-algebra structure on $\Sigma\mathrm{Hom}(B_{\geq 2}(A), \Sigma A) \otimes N^*(\Delta^2)$, and by Corollary 2.3.9, looking at the vertices of $d(\omega) + \sum_{n\geq 1} \omega\{\!\{\omega\}\!\}_n$ gives the Maurer-Cartan condition on $\phi_0, \phi_1, \phi_2 \in \mathrm{Hom}(B_{\geq 2}(A), \Sigma A)$. Looking at the components given by $\underline{01}^\vee, \underline{02}^\vee$ and $\underline{12}^\vee$ also give the Maurer-Cartan condition on the elements

$$-\phi_0 \otimes \underline{0}^\vee - \phi_1 \otimes \underline{1}^\vee - \phi_{01} \otimes \underline{01}^\vee,$$

$$-\phi_0 \otimes \underline{0}^\vee - \phi_2 \otimes \underline{1}^\vee - \phi_{02} \otimes \underline{01}^\vee,$$
$$-\phi_1 \otimes \underline{0}^\vee - \phi_2 \otimes \underline{1}^\vee - \phi_{12} \otimes \underline{01}^\vee.$$

In particular, by Proposition 2.4.21, such datas are equivalent to giving three ∞-morphisms $\Phi_{01} : (A, \phi_1) \longrightarrow (A, \phi_0), \Phi_{02} : (A, \phi_2) \longrightarrow (A, \phi_0)$ and $\Phi_{12} : (A, \phi_2) \longrightarrow (A, \phi_1)$ which reduce to the identity on ΣA. We now analyze the $\underline{012}^\vee$ component of $d(\omega) + \sum_{n\geq 1} \omega\{\!\{\omega\}\!\}_n$. By Corollary 2.3.9, the Maurer-Cartan condition on ω gives, when looking at the $\underline{012}^\vee$ component,

$$d(\phi_{012}) - \phi_{01} \overline{\circledcirc} \phi_{12} + \phi_{02} + \phi_{012}\langle\phi_2\rangle + \sum_{i,j\geq 0} \phi_0\langle\underbrace{\phi_{02}, \dots, \phi_{02}}_{i}, \phi_{012}, \underbrace{\phi_{01}\overline{\circledcirc}\phi_{12}, \dots, \phi_{01}\overline{\circledcirc}\phi_{12}}_{j}\rangle = 0.$$

Now let $\Phi_{012} : \overline{B}(A) \longrightarrow \overline{B}(A \otimes N^*(\Delta^1))$ be the unique morphism of coalgebras such that its composite with the projection on $\Sigma A \otimes N^*(\Delta^1)$ is

$$(1 + \phi_{02}) \otimes \underline{0}^\vee + \phi_{012} \otimes \underline{01}^\vee + (1 + \phi_{01}\overline{\circledcirc}\phi_{12}) \otimes \underline{1}^\vee.$$

We characterize the equation

$$\pi_{\Sigma A \otimes N^*(\Delta^1)}(d + \partial_{\phi_0})\Phi_{012} = \pi_{\Sigma A \otimes N^*(\Delta^1)}\Phi_{012}(d + \partial_{\phi_2}).$$

On one hand, we have

$$\begin{aligned} \pi_{\Sigma A \otimes N^*(\Delta^1)}(d + \partial_{\phi_0})\Phi_{012} = {} & (d_{\Sigma A} + d\phi_{02} + \phi_0 \circledcirc (1 + \phi_{02})) \otimes \underline{0}^\vee \\ & + (d_{\Sigma A} + d\phi_{01}\overline{\circledcirc}\phi_{12} + \phi_0 \circledcirc (1 + \phi_{01}\overline{\circledcirc}\phi_{12})) \otimes \underline{1}^\vee \\ + (d\phi_{012} - \phi_{01}\overline{\circledcirc}\phi_{12} + \phi_{02} + \sum_{i,j\geq 0} & \phi_0\langle\underbrace{\phi_{02}, \dots, \phi_{02}}_{i}, \phi_{012}, \underbrace{\phi_{01}\overline{\circledcirc}\phi_{12}, \dots, \phi_{01}\overline{\circledcirc}\phi_{12}}_{j}\rangle) \otimes \underline{01}^\vee. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \pi_{\Sigma A \otimes N^*(\Delta^1)}\Phi_{012}(d + \partial_{\phi_2}) = {} & (d_{\Sigma A} + \phi_{02}d + \phi_2 + \phi_{02}\langle\phi_2\rangle) \otimes \underline{0}^\vee \\ & + (d_{\Sigma A} + \phi_{01}\overline{\circledcirc}\phi_{12}d + (\phi_{01}\overline{\circledcirc}\phi_{12})\langle\phi_2\rangle) \otimes \underline{1}^\vee \\ & - (\phi_{012}d + \phi_{012}\langle\phi_2\rangle) \otimes \underline{01}^\vee, \end{aligned}$$

which proves the proposition. □

We now characterize $\mathcal{MC}_3(\mathrm{Hom}(B_{\geq 2}(A), \Sigma A))$. We first show how to compose homotopies from (A,ϕ) to $(A\otimes N^*(\Delta^1),\phi')$ for some Maurer-Cartan elements $\phi,\phi' \in \mathcal{MC}(\mathrm{Hom}(B_{\geq 2}(A),\Sigma A))$. Let $f,g,h : (A,\phi) \longrightarrow (A,\phi')$. Let $H_1 : (A,\phi) \longrightarrow (A\otimes N^*(\Delta^1),\phi')$ be a homotopy from f to g, and $H_2 : (A,\phi) \longrightarrow (A\otimes N^*(\Delta^1),\phi')$ be a homotopy from g to h. We consider the pullback

$$\begin{tikzcd}
N^*(\Delta^1) \underset{\mathbb{K}}{\times} N^*(\Delta^1) \arrow[r, dashed, "\pi_1"] \arrow[d, dashed, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & N^*(\Delta^1) \arrow[d, "d_0"] \\
N^*(\Delta^1) \arrow[r, "d_1"'] & \mathbb{K}
\end{tikzcd},$$

where we identify $N^*(\Delta^0)$ with $\mathbb{K}$. Explicitly, we have $N^*(\Delta^1) \underset{\mathbb{K}}{\times} N^*(\Delta^1) = (N^*(\Delta^1) \times N^*(\Delta^1))/((\underline{1}^\vee, 0) \sim (0, \underline{0}^\vee))$. One can see that the algebra structure of $N^*(\Delta^1)\times N^*(\Delta^1)$ preserves the equivalence relation $\sim$ so that $A \otimes (N^*(\Delta^1) \underset{\mathbb{K}}{\times} N^*(\Delta^1))$ is a path object for A in the category of A_∞-algebras. We thus obtain a homotopy $H := H_2 \times H_1 : (A,\phi) \to (A\otimes (N^*(\Delta^1) \underset{\mathbb{K}}{\times} N^*(\Delta^1)),\phi')$ from f to h.

Now let $G_1, G_2 : (A,\phi) \to (A\otimes (N^*(\Delta^1) \underset{\mathbb{K}}{\times} N^*(\Delta^1)),\phi')$ be two homotopies from f to h obtained as above. In the next proposition, we use a particular way to compose G_1 with G_2. This composition is defined as follows. Let $N^*_\square(\square^2) = N^*(\Delta^1)\otimes N^*(\Delta^1)$ and $N^*_\square(\partial\square^2) = N^*_\square(\square^2)/(\mathbb{K}\cdot \underline{01}^\vee \otimes \underline{01}^\vee)$. We consider the morphisms of algebras $\Gamma_1, \Gamma_2 : N^*_\square(\partial\square^2) \longrightarrow N^*(\Delta^1) \underset{\mathbb{K}}{\times} N^*(\Delta^1)$ defined by

$$\begin{array}{rccc}
\Gamma_1 : & N^*_\square(\partial\square^2) & \longrightarrow & N^*(\Delta^1) \underset{\mathbb{K}}{\times} N^*(\Delta^1) \\
& \underline{0}^\vee \otimes \underline{0}^\vee & \longmapsto & (\underline{0}^\vee, 0) \\
& \underline{0}^\vee \otimes \underline{1}^\vee & \longmapsto & (\underline{1}^\vee, 0) \\
& \underline{1}^\vee \otimes \underline{1}^\vee & \longmapsto & (0, \underline{1}^\vee) \\
& \underline{0}^\vee \otimes \underline{01}^\vee & \longmapsto & (\underline{01}^\vee, 0) \\
& \underline{01}^\vee \otimes \underline{1}^\vee & \longmapsto & (0, \underline{01}^\vee)
\end{array}
\quad \text{and} \quad
\begin{array}{rccc}
\Gamma_2 : & N^*_\square(\partial\square^2) & \longrightarrow & N^*(\Delta^1) \underset{\mathbb{K}}{\times} N^*(\Delta^1) \\
& \underline{0}^\vee \otimes \underline{0}^\vee & \longmapsto & (\underline{0}^\vee, 0) \\
& \underline{1}^\vee \otimes \underline{0}^\vee & \longmapsto & (\underline{1}^\vee, 0) \\
& \underline{1}^\vee \otimes \underline{1}^\vee & \longmapsto & (0, \underline{1}^\vee) \\
& \underline{01}^\vee \otimes \underline{0}^\vee & \longmapsto & (\underline{01}^\vee, 0) \\
& \underline{1}^\vee \otimes \underline{01}^\vee & \longmapsto & (0, \underline{01}^\vee)
\end{array}.$$

From a geometrical point of view, the morphism Γ_1 allows us to see the product $N^*(\Delta^1) \underset{\mathbb{K}}{\times} N^*(\Delta^1)$ as the top left corner of $N^*_\square(\square^2)$, while Γ_2 allows us to see it at the bottom right corner of $N^*_\square(\square^2)$. In particular, one can check that $N^*_\square(\partial\square^2)$ is the pullback of the diagram

$$\begin{tikzcd}
N^*_\square(\partial\square^2) \arrow[r, dashed, "\Gamma_1"] \arrow[d, dashed, "\Gamma_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & N^*(\Delta^1) \underset{\mathbb{K}}{\times} N^*(\Delta^1) \arrow[d, two heads] \\
N^*(\Delta^1) \underset{\mathbb{K}}{\times} N^*(\Delta^1) \arrow[r, two heads] & \mathbb{K}\cdot(\underline{0}^\vee, 0) \oplus \mathbb{K}\cdot(0, \underline{1}^\vee)
\end{tikzcd}.$$

Since G_1 and G_2 are homotopies from f to h, their projection on $\Sigma A \otimes \mathbb{K}\cdot(\underline{0}^\vee, 0)$ (respectively $\Sigma A\otimes \mathbb{K}\cdot(0,\underline{1}^\vee)$) agree and are given by h (respectively f). Therefore, the morphisms G_1 and G_2 induce an ∞-morphism $G_1\square G_2 : (A,\phi) \to (A \otimes N^*_{\square}(\partial\square^2), \phi')$ given by the following pullback square diagram:

$$\begin{tikzcd}
(A,\phi) \arrow[r, dashed, "G_1\square G_2"] \arrow[rr, bend left, "G_1"] \arrow[dr, "G_2"'] & (A\otimes N^*_{\square}(\partial\square^2),\phi') \arrow[r, "id\otimes\Gamma_1"] \arrow[d, "id\otimes\Gamma_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & (A\otimes(N^*(\Delta^1)\underset{\mathbb{K}}{\times}N^*(\Delta^1)),\phi') \arrow[d, two heads] \\
 & (A\otimes(N^*(\Delta^1)\underset{\mathbb{K}}{\times}N^*(\Delta^1)),\phi') \arrow[r, two heads] & (A\otimes(\mathbb{K}\cdot(\underline{0}^\vee,0)\oplus\mathbb{K}\cdot(0,\underline{1}^\vee)),\phi')
\end{tikzcd} .$$

Proposition 2.4.23. *Giving a Maurer-Cartan element in* $\mathrm{Hom}(B_{\geq 2}(A),\Sigma A)\otimes\Sigma N^*(\Delta^3)$ *is equivalent to giving* $\phi_0,\phi_1,\phi_2,\phi_3 \in \mathcal{MC}(\mathrm{Hom}(B_{\geq 2}(A),\Sigma A))$*, two homotopy diagrams of the form*

$$\begin{tikzcd}
(A,\phi_2) \arrow[r, "\Phi_{12}"] \arrow[dr, phantom, "\overset{\Phi_{123}}{\Rightarrow}"] & (A,\phi_1) \arrow[d, "\Phi_{01}"] \arrow[dl, phantom, "\overset{\Phi_{013}}{\Leftarrow}"'] \\
(A,\phi_3) \arrow[u, "\Phi_{23}"] \arrow[ur, "\Phi_{13}"] \arrow[r, "\Phi_{03}"'] & (A,\phi_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
(A,\phi_2) \arrow[r, "\Phi_{12}"] \arrow[dr, "\Phi_{02}"] \arrow[dr, phantom, "\overset{\Phi_{023}}{\Rightarrow}"'] & (A,\phi_1) \arrow[d, "\Phi_{01}"] \arrow[dl, phantom, "\overset{\Phi_{012}}{\Leftarrow}"] \\
(A,\phi_3) \arrow[u, "\Phi_{23}"] \arrow[r, "\Phi_{03}"'] & (A,\phi_0)
\end{tikzcd}$$

and a lifting diagram

$$\begin{tikzcd}
 & (A\otimes N^*_{\square}(\square^2),\phi_0) \arrow[d, two heads] \\
(A,\phi_3) \arrow[ur, dashed, "\exists\Phi_{0123}"] \arrow[r, "H_1\square H_2"'] & (A\otimes N^*_{\square}(\partial\square^2),\phi_0)
\end{tikzcd} ,$$

where we denote by $H_1,H_2 : (A,\phi_3) \longrightarrow (A\otimes(N^*(\Delta^1)\underset{\mathbb{K}}{\times}N^*(\Delta^1)),\phi_0)$ *the homotopies from* $\Phi_{01}\Phi_{12}\Phi_{23}$ *to* Φ_{03} *given by the homotopy diagrams.*

Proof. Let

$$\begin{aligned}
\omega := & -\phi_0\otimes\underline{0}^\vee - \phi_1\otimes\underline{1}^\vee - \phi_2\otimes\underline{2}^\vee - \phi_3\otimes\underline{3}^\vee \\
& - \phi_{01}\otimes\underline{01}^\vee - \phi_{02}\otimes\underline{02}^\vee - \phi_{12}\otimes\underline{12}^\vee - \phi_{03}\otimes\underline{03}^\vee - \phi_{13}\otimes\underline{13}^\vee - \phi_{23}\otimes\underline{23}^\vee \\
& - \phi_{012}\otimes\underline{012}^\vee - \phi_{013}\otimes\underline{013}^\vee - \phi_{023}\otimes\underline{023}^\vee - \phi_{123}\otimes\underline{123}^\vee - \phi_{0123}\otimes\underline{0123}^\vee
\end{aligned}$$

be an element of $\mathrm{Hom}(B_{\geq 2}(A),\Sigma A)\otimes\Sigma N^*(\Delta^3)$. By Proposition 2.4.22, the Maurer-Cartan condition on the four faces of ω is precisely equivalent to giving the first two diagrams given in the assertion of the proposition, since the $\Gamma\Lambda\mathcal{PL}_\infty$-algebra structure of $\mathrm{Hom}(B_{\geq 2}(A),\Sigma A)\otimes N^*(\Delta^3)$ is compatible with the simplicial structures. From now on, we suppose that $d_0\omega, d_1\omega, d_2\omega$ and $d_3\omega$ are elements of $\mathcal{MC}_2(\mathrm{Hom}(B_{\geq 2}(A),\Sigma A))$.

Then the only possibly non-zero component of $d(\omega) + \sum_{n\geq 1} \omega\{\!\{\omega\}\!\}_n$ is the $\underline{0123}^\vee$ component. By Corollary 2.3.9, this component is 0 if and only if we have the identity

$$\begin{aligned}
&d(\phi_{0123}) + \phi_{023} - \phi_{123} + \phi_{013} - \phi_{012} - \phi_{0123}\langle \phi_3 \rangle \\
&\quad + \sum_{k\geq 1} \phi_{012}\langle \underbrace{\phi_{23}, \dots, \phi_{23}}_{k} \rangle - \sum_{i,j\geq 0} \phi_{01}\langle \underbrace{\phi_{13}, \dots, \phi_{13}}_{i}, \phi_{123}, \underbrace{\phi_{12}\overline{\circledcirc}\phi_{23}, \dots, \phi_{12}\overline{\circledcirc}\phi_{23}}_{j} \rangle \\
&+ \sum_{i,j,k\geq 0} \phi_0\langle \underbrace{\phi_{03}, \dots, \phi_{03}}_{i}, \phi_{023}, \underbrace{\phi_{02}\overline{\circledcirc}\phi_{23}}_{j}, \phi_{012} \circledcirc (1 + \phi_{23}), \underbrace{\phi_{01}\overline{\circledcirc}\phi_{12}\overline{\circledcirc}\phi_{23}, \dots, \phi_{01}\overline{\circledcirc}\phi_{12}\overline{\circledcirc}\phi_{23}}_{m} \rangle \\
&+ \sum_{i,j,m\geq 0} \phi_0\langle \underbrace{\phi_{03}, \dots, \phi_{03}}_{i}, \phi_{013}, \underbrace{\phi_{01}\overline{\circledcirc}\phi_{13}, \dots, \phi_{01}\overline{\circledcirc}\phi_{13}}_{j}, \phi_{123}, \underbrace{\phi_{01}\overline{\circledcirc}\phi_{12}\overline{\circledcirc}\phi_{23}, \dots, \phi_{01}\overline{\circledcirc}\phi_{12}\overline{\circledcirc}\phi_{23}}_{k} \rangle \\
&\qquad + \sum_{i,j,k,l,m\geq 0} \phi_0\langle \underbrace{\phi_{03}, \dots, \phi_{03}}_{i}, \phi_{013}, \underbrace{\phi_{01}\overline{\circledcirc}\phi_{13}, \dots, \phi_{01}\overline{\circledcirc}\phi_{13}}_{j}, \\
&\quad \phi_{01}\langle \underbrace{\phi_{12}, \dots, \phi_{12}}_{k}, \phi_{123}, \underbrace{\phi_{12}\overline{\circledcirc}\phi_{23}, \dots, \phi_{12}\overline{\circledcirc}\phi_{23}}_{l} \rangle, \underbrace{\phi_{01}\overline{\circledcirc}\phi_{12}\overline{\circledcirc}\phi_{23}, \dots, \phi_{01}\overline{\circledcirc}\phi_{12}\overline{\circledcirc}\phi_{23}}_{m} \rangle \\
&\qquad + \sum_{i,j\geq 0} \phi_0\langle \underbrace{\phi_{03}, \dots, \phi_{03}}_{i}, \phi_{0123}, \underbrace{\phi_{01}\overline{\circledcirc}\phi_{12}\overline{\circledcirc}\phi_{23}, \dots, \phi_{01}\overline{\circledcirc}\phi_{12}\overline{\circledcirc}\phi_{23}}_{j} \rangle = 0.
\end{aligned}$$

We let $\Phi_{0123} : \overline{B}(A) \longrightarrow \overline{B}(A \otimes N^*_\square(\square^2))$ to be the morphism of coalgebras whose composite with the projection on $\Sigma A \otimes N^*_\square(\square^2)$ is

$$\begin{aligned}
&(1 + \phi_{03}) \otimes \underline{0}^\vee \otimes \underline{0}^\vee \\
&\qquad + (1 + \phi_{01}) \circledcirc (1 + \phi_{13}) \otimes \underline{0}^\vee \otimes \underline{1}^\vee \\
&\qquad + (1 + \phi_{02}) \circledcirc (1 + \phi_{23}) \otimes \underline{1}^\vee \otimes \underline{0}^\vee \\
&\qquad (1 + \phi_{01}) \circledcirc (1 + \phi_{12}) \circledcirc (1 + \phi_{23}) \otimes \underline{1}^\vee \otimes \underline{1}^\vee \\
&\qquad + \phi_{023} \otimes \underline{01}^\vee \otimes \underline{0}^\vee \\
&\qquad + \phi_{013} \otimes \underline{0}^\vee \otimes \underline{01}^\vee \\
&\qquad + \phi_{012} \circledcirc (1 + \phi_{23}) \otimes \underline{1}^\vee \otimes \underline{01}^\vee \\
&+ \left(\phi_{123} + \sum_{i,j\geq 0} \phi_{01}\langle \underbrace{\phi_{12}, \dots, \phi_{12}}_{i}, \phi_{123}, \underbrace{\phi_{12}\overline{\circledcirc}\phi_{23}, \dots, \phi_{12}\overline{\circledcirc}\phi_{23}}_{j} \rangle \right) \otimes \underline{01}^\vee \otimes \underline{1}^\vee \\
&\qquad + \phi_{0123} \otimes \underline{01}^\vee \otimes \underline{01}^\vee.
\end{aligned}$$

We check that ω is a Maurer-Cartan element if and only if Φ_{0123} commutes with the differentials. The latter condition is expressed by the identity

$$\pi_{\Sigma A \otimes N^*_\square(\square^2)} \Phi_{0123}(d + \partial_{\phi_3}) - \pi_{\Sigma A \otimes N^*_\square(\square^2)}(d + \partial_{\phi_0})\Phi_{0123}) = 0.$$

Since the morphisms $\Phi_{03}, \Phi_{01}\Phi_{13}, \Phi_{02}\Phi_{23}$ and $\Phi_{01}\Phi_{12}\Phi_{23}$ commute with the differentials, the components given by $\underline{0}^\vee \otimes \underline{0}^\vee, \underline{0}^\vee \otimes \underline{1}^\vee, \underline{1}^\vee \otimes \underline{0}^\vee$ and $\underline{1}^\vee \otimes \underline{1}^\vee$ are indeed 0. Since Φ_{023} and Φ_{013} commute with the differentials, the components given by $\underline{01}^\vee \otimes \underline{0}^\vee$ and $\underline{0}^\vee \otimes \underline{01}^\vee$ are also 0. We now look at the component given by $\underline{1}^\vee \otimes \underline{01}^\vee$. Since the algebra structure of $N^*_\square(\square^2)$ is compatible with its underlying simplicial structure, it is equivalent to check that the element

$$(1+\phi_{02}\overline{\circledcirc}\phi_{23})\otimes\underline{0}^\vee+\phi_{012}\circledcirc(1+\phi_{23})\otimes\underline{01}^\vee+(1+\phi_{01}\overline{\circledcirc}\phi_{12}\overline{\circledcirc}\phi_{23})\otimes\underline{1}^\vee$$

is a Maurer-Cartan element in $\mathrm{Hom}(B_{\geq 2}(A),A)\otimes N^*(\Delta^1)$. From Proposition 2.4.22, one can see that it is equivalent to check that the composite $\Phi_{012}\Phi_{23}:(\overline{B}(A),\phi_3)\longrightarrow(\overline{B}(A\otimes N^*(\Delta^1),\phi_0)$ commutes with the differentials, which is the case since Φ_{012} and Φ_{23} commute with the differentials. Analogously, we $\underline{01}^\vee\otimes\underline{1}^\vee$ is also 0, since the composite $\Phi_{01}\Phi_{123}$ commutes with the differentials.

We now look at the $\underline{01}^\vee\otimes\underline{01}^\vee$ component. The composite $\pi_{\Sigma A\otimes N^*_\square(\square^2)}\Phi_{0123}(d+\partial_{\phi_3})$ gives

$$(\phi_{0123}d+\phi_{0123}\langle\phi_3\rangle)\otimes\underline{01}^\vee\otimes\underline{01}^\vee$$

as $\underline{01}^\vee\otimes\underline{01}^\vee$ component. We now compute the $\underline{01}^\vee\otimes\underline{01}^\vee$ component given by the composite $\pi_{\Sigma A\otimes N^*_\square(\square^2)}(d+\partial_{\phi_0})\Phi_{0123}$. Computing $\pi_{\Sigma A\otimes N^*_\square(\square^2)}d\Phi_{0123}$ gives the terms

$$\begin{aligned}(d\phi_{0123}-\phi_{013}+\phi_{023}+\phi_{012}\circledcirc(1+\phi_{23})\\ -\phi_{123}-\sum_{i,j\geq 0}\phi_{01}\langle\underbrace{\phi_{12},\dots,\phi_{12}}_{i},\phi_{123},\underbrace{\phi_{12}\overline{\circledcirc}\phi_{23},\dots,\phi_{12}\overline{\circledcirc}\phi_{23}}_{j}\rangle)\otimes\underline{01}^\vee\otimes\underline{01}^\vee.\end{aligned}$$

We now compute $\pi_{\Sigma A\otimes N^*_\square(\square^2)}\partial_{\phi_0}\Phi_{0123}$. Note that the only way to write $\underline{01}^\vee\otimes\underline{01}^\vee$ as a product in $N^*(\Delta^1)\otimes N^*(\Delta^1)$ are given by one of the three following products:

$$(\underline{0}^\vee\otimes\underline{0}^\vee)\overset{i}{\cdots}(\underline{0}^\vee\otimes\underline{0}^\vee)\cdot(\underline{01}^\vee\otimes\underline{0}^\vee)\cdot(\underline{1}^\vee\otimes\underline{0}^\vee)\overset{j}{\cdots}(\underline{1}^\vee\otimes\underline{0}^\vee)\cdot(\underline{1}^\vee\otimes\underline{01}^\vee)\cdot(\underline{1}^\vee\otimes\underline{1}^\vee)\overset{k}{\cdots}(\underline{1}^\vee\otimes\underline{1}^\vee);$$

$$(\underline{0}^\vee\otimes\underline{0}^\vee)\overset{i}{\cdots}(\underline{0}^\vee\otimes\underline{0}^\vee)\cdot(\underline{0}^\vee\otimes\underline{01}^\vee)\cdot(\underline{0}^\vee\otimes\underline{1}^\vee)\overset{j}{\cdots}(\underline{0}^\vee\otimes\underline{1}^\vee)\cdot(\underline{01}^\vee\otimes\underline{1}^\vee)\cdot(\underline{1}^\vee\otimes\underline{1}^\vee)\overset{k}{\cdots}(\underline{1}^\vee\otimes\underline{1}^\vee);$$

$$(\underline{0}^\vee\otimes\underline{0}^\vee)\overset{i}{\cdots}(\underline{0}^\vee\otimes\underline{0}^\vee)\cdot(\underline{01}^\vee\otimes\underline{01}^\vee)\cdot(\underline{1}^\vee\otimes\underline{1}^\vee)\overset{j}{\cdots}(\underline{1}^\vee\otimes\underline{1}^\vee).$$

for every $i,j,k\geq 0$. These type of products give respectively

$$\begin{aligned}-\sum_{i,j,k\geq 0}\phi_0\langle\underbrace{\phi_{03},\dots,\phi_{03}}_{i},\phi_{023},\underbrace{\phi_{02}\overline{\circledcirc}\phi_{23},\dots,\phi_{02}\overline{\circledcirc}\phi_{23}}_{j},\phi_{012}\circledcirc(1+\phi_{23}),\\ \underbrace{\phi_{01}\overline{\circledcirc}\phi_{12}\overline{\circledcirc}\phi_{23},\dots,\phi_{01}\overline{\circledcirc}\phi_{12}\overline{\circledcirc}\phi_{23}}_{k}\rangle\otimes\underline{01}^\vee\otimes\underline{01}^\vee;\end{aligned}$$

$$\begin{aligned}\sum_{i,j,k,l,m\geq 0}\phi_0\langle\underbrace{\phi_{03},\dots,\phi_{03}}_{i},\phi_{013},\underbrace{\phi_{01}\overline{\circledcirc}\phi_{13},\dots,\phi_{01}\overline{\circledcirc}\phi_{13}}_{j},\\ \phi_{123}+\phi_{01}\langle\underbrace{\phi_{12},\dots,\phi_{12}}_{k},\phi_{123},\underbrace{\phi_{12}\overline{\circledcirc}\phi_{23},\dots,\phi_{12}\overline{\circledcirc}\phi_{23}}_{l}\rangle,\underbrace{\phi_{01}\overline{\circledcirc}\phi_{12}\overline{\circledcirc}\phi_{23},\dots,\phi_{01}\overline{\circledcirc}\phi_{12}\overline{\circledcirc}\phi_{23}}_{m}\rangle\otimes\underline{01}^\vee\otimes\underline{01}^\vee;\end{aligned}$$

$$\sum_{i,j\geq 0}\phi_0\langle\underbrace{\phi_{03},\dots,\phi_{03}}_{i},\phi_{0123},\underbrace{\phi_{01}\overline{\circledcirc}\phi_{12}\overline{\circledcirc}\phi_{23},\dots,\phi_{01}\overline{\circledcirc}\phi_{12}\overline{\circledcirc}\phi_{23}}_{j}\rangle\otimes\underline{01}^\vee\otimes\underline{01}^\vee,$$

as $\underline{01}^\vee\otimes\underline{01}^\vee$. We thus have obtained that Φ_{0123} commutes with the differentials if and only if ω is a Maurer-Cartan element, which proves the lemma. $\square$

2.4.4 A Goldman-Millson theorem

Our next goal is to prove an extension of the classical Goldman-Millson theorem for Lie-algebras (see [GM88, §2.4]). The proof of our analogue will be adapted from the proof given in [MR23b, §6] in the setting of associative algebras up to homotopy.

We first prove that the category $\Gamma\Lambda\mathcal{PL}_\infty$ admits finite products.

Lemma 2.4.24. *Let $(V_1, Q_{V_1}), (V_2, Q_{V_2}) \in \Gamma\Lambda\mathcal{PL}_\infty$. Then there exists a $\Gamma\Lambda\mathcal{PL}_\infty$-algebra structure on $V_1 \times V_2$ such that the morphisms $\pi_{V_1} : V_1 \times V_2 \longrightarrow V_1$ and $\pi_{V_2} : V_1 \times V_2 \longrightarrow V_2$ are strict morphisms of $\Gamma\Lambda\mathcal{PL}_\infty$-algebras.*
Moreover, for every $\phi_1 : W \rightsquigarrow V_1$ and $\phi_2 : W \rightsquigarrow V_2$, there exists a unique ∞-morphism denoted by $\phi_1 \times \phi_2 : W \rightsquigarrow V_1 \times V_2$ such that $\pi_{V_1}(\phi_1 \times \phi_2) = \phi_1$ and $\pi_{V_2}(\phi_1 \times \phi_2) = \phi_2$.

Proof. We let $Q_{V_1\times V_2}$ be the coderivation produced by the morphism

$$Q^0_{V_1\times V_2} : \Gamma\mathrm{Perm}^c(V_1 \times V_2) \twoheadrightarrow \Gamma\mathrm{Perm}^c(V_1) \times \Gamma\mathrm{Perm}^c(V_2) \xrightarrow{Q^0_{V_1}\times Q^0_{V_2}} V_1 \times V_2 \ .$$

Recall that the coderivation $Q_{V_1\times V_2}$ is obtained from $Q^0_{V_1\times V_2}$ by $Q_{V_1\times V_2} = \widetilde{\Psi}_1(Q^0_{V_1\times V_2}) + \widetilde{\Psi}_2(Q^0_{V_1\times V_2})$ (see the proof of Proposition 2.2.15). We check that $Q_{V_1\times V_2}Q_{V_1\times V_2} = 0$. By definition of $\widetilde{\Psi}_1$, we have the following commutative diagram:

$$\begin{tikzcd}
& (V_1 \times V_2)\otimes\Gamma(V_1\times V_2) \arrow[dl, two heads] \arrow[dr, "\widetilde{\Psi}_1(Q^0_{V_1\times V_2})"] & \\
(V_1\otimes\Gamma(V_1))\times(V_2\otimes\Gamma(V_2)) \arrow[d, "\widetilde{\Psi}_1(Q^0_{V_1})\times\widetilde{\Psi}_1(Q^0_{V_2})"'] & & (V_1 \times V_2)\otimes\Gamma(V_1\times V_2) \arrow[d, "Q^0_{V_1\times V_2}"] \\
(V_1\otimes\Gamma(V_1))\times(V_2\otimes\Gamma(V_2)) \arrow[rr, "Q^0_{V_1}\times Q^0_{V_2}"'] & & V_1\times V_2
\end{tikzcd} \cdot$$

We also have the commutative diagram, by definition of $\widetilde{\Psi}_2$:

$$\begin{tikzcd}
& (V_1 \times V_2)\otimes\Gamma(V_1\times V_2) \arrow[dl, two heads] \arrow[dr, "\widetilde{\Psi}_2(Q^0_{V_1\times V_2})"] & \\
(V_1\otimes\Gamma(V_1))\times(V_2\otimes\Gamma(V_2)) \arrow[d, "\widetilde{\Psi}_2(Q^0_{V_1})\times\widetilde{\Psi}_2(Q^0_{V_2})"'] & & (V_1 \times V_2)\otimes\Gamma(V_1\times V_2) \arrow[d, "Q^0_{V_1\times V_2}"] \\
(V_1\otimes\Gamma(V_1))\times(V_2\otimes\Gamma(V_2)) \arrow[rr, "Q^0_{V_1}\times Q^0_{V_2}"'] & & V_1\times V_2
\end{tikzcd} \cdot$$

Finally, we have proved that $Q^0_{V_1\times V_2}\widetilde{\Psi}(Q^0_{V_1\times V_2})$ fits in the following commutative diagram

$$
\begin{tikzcd}
 & (V_1 \times V_2) \otimes \Gamma(V_1 \times V_2) \arrow[dl, two heads] \arrow[dr, "\widetilde{\Psi}(Q^0_{V_1 \times V_2})"] & \\
(V_1 \otimes \Gamma(V_1)) \times (V_2 \otimes \Gamma(V_2)) \arrow[d, "\widetilde{\Psi}(Q^0_{V_1}) \times \widetilde{\Psi}(Q^0_{V_2})"'] \arrow[drr, "0"] & & (V_1 \times V_2) \otimes \Gamma(V_1 \times V_2) \arrow[d, "Q^0_{V_1 \times V_2}"] \\
(V_1 \otimes \Gamma(V_1)) \times (V_2 \otimes \Gamma(V_2)) \arrow[rr, "Q^0_{V_1} \times Q^0_{V_2}"'] & & V_1 \times V_2
\end{tikzcd},
$$

which proves that $(V_1 \times V_2, Q_{V_1 \times V_2}) \in \Gamma\Lambda\mathcal{PL}_\infty$. Now let $\phi_1 : W \rightsquigarrow V_1$ and $\phi_2 : W \rightsquigarrow V_2$ be two ∞-morphisms. We define $\phi : W \rightsquigarrow V_1 \times V_2$ by $\phi^0 = \phi_1^0 \times \phi_2^0$. We prove that this gives an ∞-morphism i.e. $\phi^0 Q = (Q^0_{V_1} \times Q^0_{V_2})\phi$. This can be proved with the following commutative diagram:

$$
\begin{tikzcd}
\Gamma\mathrm{Perm}^c(W) \arrow[rr, "Q"] \arrow[d, "\phi"'] & & \Gamma\mathrm{Perm}^c(W) \arrow[d, "\phi_1^0 \times \phi_2^0"] \\
\Gamma\mathrm{Perm}^c(V_1 \times V_2) \arrow[r, two heads, "\pi_{V_1} \times \pi_{V_2}"] \arrow[rr, bend right, "Q^0_{V_1 \times V_2}"'] & \Gamma\mathrm{Perm}^c(V_1) \times \Gamma\mathrm{Perm}^c(V_2) \arrow[r, "Q^0_{V_1} \times Q^0_{V_2}"] & V_1 \times V_2
\end{tikzcd}.
$$

The identities $\pi_{V_1}(\phi_1 \times \phi_2) = \phi_1$ and $\pi_{V_2}(\phi_1 \times \phi_2) = \phi_2$ follow by immediate computations. $\square$

Remark 2.4.25. *By an immediate check, the above definitions extend to the category $\widehat{\Gamma\Lambda\mathcal{PL}_\infty}$ of complete $\Gamma\Lambda\mathcal{PL}_\infty$-algebras. Explicitly, if (V_1, Q_{V_1}) and (V_2, Q_{V_2}) are complete with respect to some filtrations, then $(V_1 \times V_2, Q_{V_1 \times V_2})$ is also complete with the filtration*

$$F_n(V_1 \times V_2) = F_n V_1 \times F_n V_2.$$

In this setting, we deduce immediately from the definition of $Q^0_{V_1 \times V_2}$ that we have a bijection

$$\mathcal{MC}(V_1 \times V_2) \simeq \mathcal{MC}(V_1) \times \mathcal{MC}(V_2).$$

We give an analogue of [MR23b, Proposition 5.2].

Lemma 2.4.26. *Let $A, B \in \mathrm{dg}\widehat{\mathrm{Mod}}_{\mathbb{K}}$ be such that $A \otimes \Sigma N^*(\Delta^\bullet)$ and $B \otimes \Sigma N^*(\Delta^\bullet)$ are endowed with the structure of simplicial $\widehat{\Gamma\Lambda\mathcal{PL}_\infty}$-algebras. Let $\Theta : A \longrightarrow B$ be a morphism in $\mathrm{dg}\widehat{\mathrm{Mod}}_{\mathbb{K}}$ such that $\Theta \otimes id$ is a strict morphism of $\widehat{\Gamma\Lambda\mathcal{PL}_\infty}$-algebras for every $n \geq 0$. Suppose that Θ is an acyclic fibration of dg $\mathbb{K}$-modules.*
Then the map

$$\Theta \otimes id : \mathcal{MC}(A \otimes \Sigma N^*(\Delta^\bullet)) \longrightarrow \mathcal{MC}(B \otimes \Sigma N^*(\Delta^\bullet))$$

is a weak equivalence of simplicial sets.

Proof. Since the two simplicial sets $\mathcal{MC}(A \otimes \Sigma N^*(\Delta^\bullet))$ and $\mathcal{MC}(B \otimes \Sigma N^*(\Delta^\bullet))$ are Kan complexes by Theorem 2.4.5, it suffices to show that $\Theta \otimes id$ induces a bijection on the sets of connected components and an isomorphism on every homotopy groups. Let $\tau : B \longrightarrow A$ and $h : A \longrightarrow A$ be such that

$$\Theta\tau = id \ ; \ id - \tau\Theta = dh + hd.$$

We endow $\mathrm{Ker}(\Theta)$ with the brace algebra structure defined by $a\langle b_1, \ldots, b_r\rangle = 0$ for every $r \geq 1$ and $a, b_1, \ldots, b_r \in \mathrm{Ker}(\Theta)$. Our first goal is to define a morphism $\Psi_\bullet : A \otimes \Sigma N^*(\Delta^\bullet) \rightsquigarrow \mathrm{Ker}(\Theta) \otimes \Sigma N^*(\Delta^\bullet)$ of simplicial $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebras. For every $n \geq 0$, let $(\Psi_n)_0^0 = (id - \tau\Theta) \otimes id$. We set, for every $k \neq 0$,

$$(\Psi_n)_k^0 = (\Psi_n)_0^0(h \otimes id)Q_k^0,$$

where we denote by Q the coderivation on $\Gamma\mathrm{Perm}^c(A \otimes \Sigma N^*(\Delta^n))$ given by the $\Gamma\Lambda\mathcal{PL}_\infty$-algebra structure on $A \otimes \Sigma N^*(\Delta^n)$. We check that $\Psi_n : A \otimes \Sigma N^*(\Delta^n) \rightsquigarrow \mathrm{Ker}(\Theta) \otimes \Sigma N^*(\Delta^n)$:

$$\begin{array}{rcl} d(\Psi_n)_k^0 & = & d(\Psi_n)_0^0(h \otimes id)Q_k^0 \\ & = & (\Psi_n)_0^0 d(h \otimes id)Q_k^0 \\ & = & (\Psi_n)_0^0 Q_k^0 - (\Psi_n)_0^0(h \otimes id)dQ_k^0 \\ & = & (\Psi_n)_0^0 Q_k^0 + \displaystyle\sum_{i=1}^{k} (\Psi_n)_0^0(h \otimes id)Q_i^0 Q_k^i \\ & = & (\Psi_n)_0^0 Q_k^0 + \displaystyle\sum_{i=1}^{k} (\Psi_n)_i^0 Q_k^i \\ & = & \displaystyle\sum_{i=0}^{k} (\Psi_n)_i^0 Q_k^i, \end{array}$$

which proves that $\Psi_n : A \otimes \Sigma N^*(\Delta^n) \rightsquigarrow \mathrm{Ker}(\Theta) \otimes \Sigma N^*(\Delta^n)$. Since $\Psi_\bullet$ is defined in terms of morphisms which are compatible with the simplicial structure, the morphism $\Psi_\bullet$ is a morphism of simplicial $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebras. Consider now, for every $n \geq 0$, the $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebra structure on $(B \times \mathrm{Ker}(\Theta)) \otimes \Sigma N^*(\Delta^\bullet)$ given by the isomorphism

$$(B \times \mathrm{Ker}(\Theta)) \otimes \Sigma N^*(\Delta^\bullet) \simeq (B \otimes \Sigma N^*(\Delta^\bullet)) \times (\mathrm{Ker}(\Theta) \otimes \Sigma N^*(\Delta^\bullet)).$$

Let $g_n = (\Theta \otimes id) \times \Psi_n : A \otimes \Sigma N^*(\Delta^n) \rightsquigarrow (B \times \mathrm{Ker}(\Theta)) \otimes \Sigma N^*(\Delta^n)$. Then $g_\bullet$ is a morphism of simplicial $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebras. We also have that $\mathcal{MC}(g_\bullet)$ is an isomorphism of simplicial sets. Indeed, we have that $(g_\bullet)_0^0$ is an isomorphism of simplicial sets, with as inverse $(b, k) \otimes \underline{x} \longmapsto (\tau(b) + k) \otimes \underline{x}$, for every $b \in B, k \in \mathrm{Ker}(\Theta)$ and $\underline{x} \in \Sigma N^*(\Delta^\bullet)$. By looking at the Maurer-Cartan spaces degree wise, we obtain the following commutative diagram:

$$\begin{tikzcd} \mathcal{MC}(A \otimes \Sigma N^*(\Delta^\bullet)) \arrow[r, "\mathcal{MC}(g_\bullet)", "\simeq"'] \arrow[d, "\mathcal{MC}(\Theta\otimes id)"'] & \mathcal{MC}((B \times \mathrm{Ker}(\Theta)) \otimes \Sigma N^*(\Delta^\bullet)) \arrow[d, "\simeq"] \\ \mathcal{MC}(B \otimes \Sigma N^*(\Delta^\bullet)) & \mathcal{MC}(B \otimes \Sigma N^*(\Delta^\bullet)) \times \mathcal{MC}(\mathrm{Ker}(\Theta) \otimes \Sigma N^*(\Delta^\bullet)) \arrow[l, two heads] \end{tikzcd}$$

It is then sufficient to prove that the projection $\mathcal{MC}(B \otimes \Sigma N^*(\Delta^\bullet)) \times \mathcal{MC}(\mathrm{Ker}(\Theta) \otimes$

$\Sigma N^*(\Delta^\bullet)) \longrightarrow \mathcal{MC}(B\otimes\Sigma N^*(\Delta^\bullet))$ is a weak equivalence of simplicial sets, which is true because $\mathcal{MC}(\mathrm{Ker}(\Theta)\otimes\Sigma N^*(\Delta^\bullet)) = \mathcal{MC}_\bullet(\mathrm{Ker}(\Theta))$, and this simplicial set has trivial π_0 and homotopy groups according to the computations of the connected components and the homotopy groups made in §2.4.2. We then have the result. □

The next lemma is an analogue of [MR23b, Proposition 5.5].

Lemma 2.4.27. *Let A, B, C be $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebras. Let $\Theta : A \rightsquigarrow C$ and $\Phi : B \rightsquigarrow C$ be two ∞-morphisms of $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebras. We suppose that Φ is strict, and that Φ_0^0 is surjective. Then there exists a $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebra structure on $A\times\mathrm{Ker}(\Phi_0^0)$ and $H : A\times\mathrm{Ker}(\Phi_0^0) \rightsquigarrow A\times B$ such that the following diagram is a pullback square diagram in $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$:*

$$\begin{tikzcd} A\times\mathrm{Ker}(\Phi_0^0) \arrow[r, dashed, "\pi_B H"] \arrow[d, dashed, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, dashed, "\Phi"] \\ A \arrow[r, dashed, "\Theta"'] & C \end{tikzcd} .$$

Proof. We follow the proof of [MR23b, Proposition 5.5]. Let $\sigma : C \longrightarrow B$ be a morphism of $\mathbb{K}$-modules such that $\Phi_0^0\sigma = id$. We define two morphisms $J_0^0, H_0^0 : A\times B \longrightarrow A\times B$ by $J_0^0(a,b) = (a, b-\sigma\Theta_0^0(a))$ and $H_0^0(a,b) = (a, \sigma\Theta_0^0(a)+b)$. We set

$$H_n^0 = (0, \sigma\Theta_n^0\pi_A);$$

$$J_n^0 = (0, -\sigma\Theta_n^0\pi_A),$$

An immediate computation gives $HJ = JH = id$. Therefore, if we denote by Q the $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebra structure on $A\times B$, then $\widetilde{Q} = JQH$ is a degree -1 coderivation on $\Gamma\mathrm{Perm}^c(A\times B)$. We note that $\widetilde{Q}$ preserves $\Gamma\mathrm{Perm}^c(A\times\mathrm{Ker}(\Phi_0^0))$ and the filtrations, so that $A\times\mathrm{Ker}(\Phi_0^0)$ is a $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebra such that $H : A\times\mathrm{Ker}(\Phi_0^0) \rightsquigarrow A\times B$. Consider now a diagram of ∞-morphisms:

$$\begin{tikzcd} D \arrow[r, "\Psi_2"] \arrow[d, "\Psi_1"'] & B \arrow[d, "\Phi"] \\ A \arrow[r, "\Theta"'] & C \end{tikzcd} .$$

Then, we have the commutative diagram

$$\begin{tikzcd} D \arrow[dr, "J(\Psi_1\times\Psi_2)"] \arrow[drr, bend left, "\Psi_2"] \arrow[ddr, bend right, "\Psi_1"'] & & \\ & A\times\mathrm{Ker}(\Phi_0^0) \arrow[r, "\pi_B H"] \arrow[d, "\pi_A"'] & B \arrow[d, "\Phi"] \\ & A \arrow[r, "\Theta"'] & C \end{tikzcd} ,$$

which proves the result. $\square$

We now prove Theorem F.

Theorem 2.4.28. *Let $\Theta : A \longrightarrow B$ be a morphism of complete brace algebras such that Θ is a weak equivalence in* $\mathrm{dgMod}_{\mathbb{K}}$. *Then $\mathcal{MC}_\bullet(\Theta) : \mathcal{MC}_\bullet(A) \longrightarrow \mathcal{MC}_\bullet(B)$ is a weak equivalence.*

Before giving the proof, note that if B is a complete brace algebra, and if we set $B^I = B \otimes N^*(\Delta^1)$, then we have the following decomposition of the diagonal map in the category of $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebras

$$\begin{tikzcd}
B \otimes \Sigma N^*(\Delta^n) \arrow[r, tail, "s_0 \otimes id"'] \arrow[rr, bend left, "\Delta"] & B^I \otimes \Sigma N^*(\Delta^n) \arrow[r, two heads, "(d_0 \times d_1) \otimes id"'] & (B \times B) \otimes \Sigma N^*(\Delta^n)
\end{tikzcd}$$

for every $n \geq 0$. This decomposition comes from Proposition 2.1.33 with $\mathcal{P} = \Lambda\mathcal{B}race$ and $R = B \otimes \Sigma N^*(\Delta^n)$. The map $s_0 : B \longrightarrow B^I$ is given by the simplicial map $s_0 : N^*(\Delta^0) \longrightarrow N^*(\Delta^1)$ and the maps $d_0, d_1 : B^I \longrightarrow B$ are given by $d_0, d_1 : N^*(\Delta^1) \longrightarrow N^*(\Delta^0)$. In particular, the morphisms s_0, d_0 and d_1 induce strict morphisms of $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebras, since the action of $\mathcal{E}$ on $N^*(\Delta^n)$ is compatible with its underlying simplicial structure.

Proof. Lemma 2.4.26 proves the theorem in the case of an acyclic fibration Θ. Consider now the general case. Since $d_0 \otimes id : B^I \otimes \Sigma N^*(\Delta^n) \longrightarrow B \otimes \Sigma N^*(\Delta^n)$ is a strict morphism and surjective, we can apply Lemma 2.4.27:

$$\begin{tikzcd}
A \otimes \Sigma N^*(\Delta^n) \arrow[r, dashed, "\exists \Psi_n"] \arrow[rr, bend left, "(s_0\Theta) \otimes id"] \arrow[dr, equal, "="'] & (A \times \mathrm{Ker}(d_0)) \otimes \Sigma N^*(\Delta^n) \arrow[r, "(\pi_{\mathrm{Ker}(d_0)} \otimes id) H_n"] \arrow[d, two heads, "\pi_A \otimes id"'] \arrow[dr, phantom, "\lrcorner", very near start] & B^I \otimes \Sigma N^*(\Delta^n) \arrow[d, two heads, "d_0 \otimes id"] \\
 & A \otimes \Sigma N^*(\Delta^n) \arrow[r, "\Theta \otimes id"'] & B \otimes \Sigma N^*(\Delta^n)
\end{tikzcd}$$

where $H_n : (A \times \mathrm{Ker}(d_0)) \otimes \Sigma N^*(\Delta^n) \rightsquigarrow (A \times B^I) \otimes \Sigma N^*(\Delta^n)$ is given by Lemma 2.4.27, and $\Psi_n : A \otimes \Sigma N^*(\Delta^n) \rightsquigarrow (A \times \mathrm{Ker}(d_0)) \otimes \Sigma N^*(\Delta^n)$ is the unique ∞-morphism which makes the previous diagram commutative.

We recall the $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebra structure on $(A \times \mathrm{Ker}(d_0)) \otimes \Sigma N^*(\Delta^n)$. Let Q_n be the coderivation on $\Gamma\mathrm{Perm}^c((A \times B^I) \otimes \Sigma N^*(\Delta^n))$ given by the product

$$(A \times B^I) \otimes \Sigma N^*(\Delta^n) \simeq (A \otimes \Sigma N^*(\Delta^n)) \times (B^I \otimes \Sigma N^*(\Delta^n)).$$

Consider the morphisms $H_n, J_n : (A \times B^I) \otimes \Sigma N^*(\Delta^n) \rightsquigarrow (A \times B^I) \otimes \Sigma N^*(\Delta^n)$ defined in the proof of Lemma 2.4.27. We note that these morphisms are strict, as the morphism $\Theta \otimes id : A \otimes \Sigma N^*(\Delta^n) \longrightarrow B \otimes \Sigma N^*(\Delta^n)$ is strict, and are defined by

$$(H_n)_0^0((a, b) \otimes \underline{x}) = (a, \sigma\Theta(a) + b) \otimes \underline{x};$$

$$(J_n)_0^0((a,b)\otimes \underline{x}) = (a, b-\sigma\Theta(a))\otimes \underline{x},$$

for every $a \in A, b \in B$ and $\underline{x} \in \Sigma N^*(\Delta^n)$, and where $\sigma : B \longrightarrow B^I$ is a splitting of $d_0 : B^I \longrightarrow B$. Then, the $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebra structure on $(A \times \mathrm{Ker}(d_0)) \otimes \Sigma N^*(\Delta^n)$ is given by

$$(\widetilde{Q}_n)_p^q = (J_n)_q^q (Q_n)_p^q (H_n)_p^p.$$

We thus see that the $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebra structures on $(A \times \mathrm{Ker}(d_0)) \otimes \Sigma N^*(\Delta^n)$ for all $n \geq 0$ endow the simplicial set $(A \times \mathrm{Ker}(d_0)) \otimes \Sigma N^*(\Delta^\bullet)$ with the structure of a strict simplicial object in $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$. Moreover, the map $\pi_A : A \times \mathrm{Ker}(d_0) \longrightarrow A$ is an acyclic fibration, and a simple computation shows that $\pi_A \otimes id$ is a strict morphism. By Lemma 2.4.26, we deduce that $\mathcal{MC}(\pi_A \otimes id)$ is a weak equivalence. By the 2 out of 3 axiom in sSet, we also have that $\mathcal{MC}(\Psi_\bullet)$ is a weak equivalence of simplicial sets.

Let $h : A \times B^I \longrightarrow A \times B^I$ be the morphism such that $(H_n)_0^0 = h \otimes id$. For every $n \geq 0$, we set

$$P_n = d_1 \pi_{B^I} h \otimes id : (A \times \mathrm{Ker}(d_0)) \otimes \Sigma N^*(\Delta^n) \longrightarrow B \otimes \Sigma N^*(\Delta^n)$$

We show that P_n is a strict acyclic fibration. First, for every $n \geq 0$, the morphism P_n is strict as it is the composite of strict morphisms. Moreover, we have the identity $\Theta \otimes id = \Psi_\bullet P_\bullet$, which shows that P_n is acyclic for every $n \geq 0$. We now prove that P_n is surjective. For every $b \in B$ and $\underline{x} \in \Sigma N^*(\Delta^n)$, we have $P_n(0, b \otimes \underline{0}^\vee \otimes \underline{x}) = b \otimes \underline{x}$ which proves that P_n is surjective for every $n \geq 0$. By Lemma 2.4.26, we have that $\mathcal{MC}(P_\bullet)$ is a weak equivalence. Finally, since we have $\Theta \otimes id = \Psi_\bullet P_\bullet$, it follows that $\mathcal{MC}(\Theta \otimes id)$ is also a weak equivalence, which proves the theorem. □

2.4.5 Comparison with the deformation theory of shifted $\mathcal{L}ie_\infty$-algebras

Let $\mathcal{L}ie_\infty$ be an operad which encodes Lie algebras up to homotopy, for instance $\mathcal{L}ie_\infty = B^c(\Lambda^{-1}\mathcal{C}om^\vee)$. We call a $\Lambda\mathcal{L}_\infty$-algebra any algebra over the operad $\Lambda\mathcal{L}ie_\infty$. These algebras have been widely studied in the literature. Recall (for instance from [DR15a, §2], or [Ber15] for the non-shifted analogue) that giving a $\Lambda\mathcal{L}_\infty$-algebra structure on a graded $\mathbb{K}$-module V is equivalent to giving degree -1 brackets

$$\underbrace{[-,\ldots,-]}_{n} : (V^{\otimes n})_{\Sigma_n} \longrightarrow V$$

for every $n \geq 0$ such that we have the higher Jacobi relations:

$$\sum_{k=1}^{n} \sum_{\sigma \in \mathrm{Sh}(k,n-k)} \pm[[x_{\sigma(1)},\ldots,x_{\sigma(k)}], x_{\sigma(k+1)},\ldots,x_{\sigma(n)}] = 0$$

for every $x_1,\ldots,x_n \in V$. In particular, the 0-bracket $d := [-]$ is a differential.

Proposition 2.4.29. *There exists an operad morphism $\mathcal{L}ie_\infty \longrightarrow \mathcal{P}re\mathcal{L}ie_\infty$ which fits in the following commutative diagram:*

$$\begin{tikzcd} \mathcal{L}ie_\infty \arrow[r] \arrow[d] & \mathcal{P}re\mathcal{L}ie_\infty \arrow[d] \\ \mathcal{L}ie \arrow[r] & \mathcal{P}re\mathcal{L}ie \end{tikzcd}.$$

In particular, every $\Lambda\mathcal{PL}_\infty$-algebra is a $\Lambda\mathcal{L}_\infty$-algebra with the brackets

$$[x_1,\dots,x_n] = \sum_{i=1}^n \pm x_i\{\!\{x_1,\dots,\widehat{x_i},\dots,x_n\}\!\}.$$

Proof. We have an operad morphism Perm $\longrightarrow \mathcal{C}om$ defined by $e_i^n \longrightarrow 1$ for every $n \geq 1$ and $1 \leq i \leq n$. By duzalization, this gives a cooperad morphism $\mathcal{C}om^\vee \longrightarrow \mathrm{Perm}^\vee$ defined by $1 \longmapsto \sum_{i=1}^n (e_i^n)^\vee$ for every $n \geq 1$. Taking the cobar construction then gives a well-defined morphism $\mathcal{L}ie_\infty \longrightarrow \mathcal{P}re\mathcal{L}ie_\infty$. The commutativity of the square comes from immediate computation. The relation between the $\Lambda\mathcal{PL}_\infty$-algebra structure and its induced $\Lambda\mathcal{L}_\infty$-algebra structure comes from the morphism $\mathcal{C}om^\vee \longrightarrow \mathrm{Perm}^\vee$. □

Proposition 2.4.29 implies that every complete $\Lambda\mathcal{PL}_\infty$-algebra is endowed with the structure of a complete $\Lambda\mathcal{L}_\infty$-algebra. For every $\Lambda\mathcal{PL}_\infty$-algebra A, we denote by $L(A)$ the underlying $\Lambda\mathcal{L}_\infty$-algebra structure on A.

From now, we work over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$. Using Theorem 2.3.11, we can use the deformation theory developed in [Rog23] for $\Lambda\mathcal{L}_\infty$-algebras (called $L[1]_\infty$-algebras in this reference). Following [Rog23, §5.6], for every Lie algebra L, we set

$$\mathcal{MC}_\bullet(L) = \mathcal{MC}(L\widehat{\otimes}\Sigma\Omega^*(\Delta^\bullet)),$$

where $\Omega^*(\Delta^n)$ denotes the dg associative and commutative algebra of polynomial De Rham forms on the simplex Δ^n, and where we consider, on the right hand-side, the Maurer-Cartan set of the $\Lambda\mathcal{L}_\infty$-algebra $L \otimes \Sigma\Omega^*(\Delta^\bullet)$ (see [Rog23, §5.4]).

Note that since $char(\mathbb{K}) = 0$, the category of $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebras is equivalent to the category of $\mathcal{P}re\mathcal{L}ie_\infty$-algebras, so that $\Gamma\Lambda\mathcal{PL}_\infty = \Lambda\mathcal{PL}_\infty$. The goal of this subsection is to prove that, for every complete brace algebra A, the simplicial sets $\mathcal{MC}_\bullet(A)$ and $\mathcal{MC}_\bullet(L(A))$ are weakly equivalent.

In the following, we distinguish the $\Lambda\mathcal{PL}_\infty$-algebra structure with the $\Lambda\mathcal{L}_\infty$-algebra structure. More precisely, for every complete $\Lambda\mathcal{PL}_\infty$-algebra V, we set:

$$\mathcal{MC}^{\Lambda\mathcal{PL}_\infty}(V) = \mathcal{MC}(V)\ ;\ \mathcal{MC}^{\Lambda\mathcal{L}_\infty}(V) = \mathcal{MC}(L(V)).$$

We also set, for every complete brace algebra A,

$$\mathcal{MC}_\bullet^{\Lambda\mathcal{PL}_\infty}(A) = \mathcal{MC}_\bullet(A)\ ;\ \mathcal{MC}_\bullet^{\Lambda\mathcal{L}_\infty}(A) = \mathcal{MC}_\bullet(L(A)),$$

where $L(A)$ is the Lie algebra endowed with the bracket $[x,y] = x\langle y\rangle - (-1)^{|x||y|}y\langle x\rangle$.

Lemma 2.4.30. *Let V be a $\widehat{\Lambda\mathcal{PL}_\infty}$-algebra. Then*

$$\mathcal{MC}^{\Lambda\mathcal{L}_\infty}(V) = \mathcal{MC}^{\Lambda\mathcal{PL}_\infty}(V).$$

Proof. By Proposition 2.4.29, we have, for every $x_1, \ldots, x_n \in V$,

$$[x_1, \ldots, x_n] = \sum_{k=1}^{n} \pm x_k \{\!\{x_1, \ldots, \widehat{x_k}, \ldots, x_n\}\!\}.$$

Then, the Maurer-Cartan equation

$$d(x) + \sum_{n\geq 1} \frac{1}{n!} x\{\!\{\underbrace{x, \ldots, x}_{n}\}\!\} = 0$$

is equivalent to the equation

$$d(x) + \sum_{n\geq 2} \frac{1}{n!} [\underbrace{x, \ldots, x}_{n}] = 0,$$

which is precisely the Maurer-Cartan equation in complete $\Lambda\mathcal{L}ie_\infty$-algebras. □

Let A be a complete brace algebra, B a dg commutative and associative algebra, and E be a $\mathcal{E}$-algebra. Then the tensor products $(A\widehat{\otimes}B)\widehat{\otimes}\Sigma E$ and $(A\widehat{\otimes}E)\widehat{\otimes}\Sigma B$ are endowed with a complete $\Lambda\mathcal{L}_\infty$-algebra structure. Indeed, the first one is induced by the composite

$$\Lambda\mathcal{L}ie_\infty \longrightarrow \Lambda\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E} \xrightarrow{\simeq} (\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{C}om) \underset{\mathrm{H}}{\otimes} \Lambda\mathcal{E} \ ,$$

while the second one is induced by the composite

$$\Lambda\mathcal{L}ie_\infty \longrightarrow \Lambda\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E} \xrightarrow{\simeq} (\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}) \underset{\mathrm{H}}{\otimes} \Lambda\mathcal{C}om \ .$$

Lemma 2.4.31. *The isomorphism*

$$(A\widehat{\otimes}B)\widehat{\otimes}\Sigma E \xrightarrow{\simeq} (A\widehat{\otimes}E)\widehat{\otimes}\Sigma B$$

which exchanges E and B is an isomorphism of complete $\Lambda\mathcal{L}_\infty$-algebras.

Proof. Straightforward computations. □

We thus obtain the following theorem.

Theorem 2.4.32. *Let A be a complete brace algebra. Then there exists a simplicial set $\mathcal{S}^A_\bullet$ and a zig-zag of weak equivalences in simplicial sets:*

$$\mathcal{MC}^{\Lambda\mathcal{PL}_\infty}_\bullet(A) \xrightarrow{\sim} \mathcal{S}^A_\bullet \xleftarrow{\sim} \mathcal{MC}^{\Lambda\mathcal{L}_\infty}_\bullet(A) \ .$$

One major consequence of this theorem is that the homotopy groups that we have computed are isomorphic to the one's found in [Ber15] if the field is of characteristic 0.

Proof. We first remark that, for any $n \geq 0$, we have a morphism of complete brace algebras which is a weak equivalence:

$$A \xrightarrow{\sim} A\widehat{\otimes}\Omega^*(\Delta^n).$$

By Theorem 2.4.28, we obtain a weak equivalence

$$\mathcal{MC}_\bullet^{\Lambda\mathcal{PL}_\infty}(A) \xrightarrow{\sim} \mathcal{MC}_\bullet^{\Lambda\mathcal{PL}_\infty}(A\widehat{\otimes}\Omega^*(\Delta^n)) = \mathcal{MC}^{\Lambda\mathcal{PL}_\infty}((A\widehat{\otimes}\Omega^*(\Delta^n))\widehat{\otimes}\Sigma N^*(\Delta^\bullet)).$$

We now apply [GJ09, Chapter IV, Proposition 1.9]. Recall that the *diagonal* of a bisimplicial set X (see [GJ09, Chapter IV, §1]) is the simplicial set $Diag(X)$ defined by

$$Diag(X)_n = X_{nn}$$

for every $n \geq 0$. Since we have a point-wise weak equivalence, this extends to the following weak-equivalence of simplicial sets

$$\mathcal{MC}_\bullet^{\Lambda\mathcal{PL}_\infty}(A) \xrightarrow{\sim} Diag(\mathcal{MC}^{\Lambda\mathcal{PL}_\infty}((A\widehat{\otimes}\Omega^*(\Delta^\bullet))\widehat{\otimes}\Sigma N^*(\Delta^\bullet))) \ ,$$

Similarly, by [DR15b, Theorem 1.1], we have a weak equivalence

$$Diag(\mathcal{MC}^{\Lambda\mathcal{L}_\infty}((A \otimes N^*(\Delta^\bullet)) \otimes \Sigma\Omega^*(\Delta^\bullet))) \xleftarrow{\sim} \mathcal{MC}_\bullet^{\Lambda\mathcal{L}_\infty}(A) \ .$$

By combining the above weak equivalences with the two previous lemmas, we obtain the following diagram:

$$\begin{tikzcd}
\mathcal{MC}_\bullet^{\Lambda\mathcal{PL}_\infty}(A) \arrow[r, "\sim"] & Diag(\mathcal{MC}^{\Lambda\mathcal{PL}_\infty}((A \otimes \Omega^*(\Delta^\bullet)) \otimes \Sigma N^*(\Delta^\bullet))) \arrow[d, dash, "="] & \\
& Diag(\mathcal{MC}^{\Lambda\mathcal{L}_\infty}((A \otimes \Omega^*(\Delta^\bullet)) \otimes \Sigma N^*(\Delta^\bullet))) \arrow[d, "\simeq"] & \\
& Diag(\mathcal{MC}^{\Lambda\mathcal{L}_\infty}((A \otimes N^*(\Delta^\bullet)) \otimes \Sigma\Omega^*(\Delta^\bullet))) & \mathcal{MC}_\bullet^{\Lambda\mathcal{L}_\infty}(L(A)) \arrow[l, "\sim"']
\end{tikzcd},$$

which proves the theorem. □

2.5 A mapping space in the category of non-symmetric operads

In this section, we give an explicit construction of a mapping space $\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}), \mathcal{P})$ in the category of non symmetric operads in terms of $\Gamma\Lambda\mathcal{PL}_\infty$ operations. Explicitly, we give a construction of a mapping space as the simplicial Maurer-Cartan set associated to the complete brace algebra $\mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})$.

In §2.5.1, we make recollections on the construction of the free operad functor and on the model structure used for operads in this memoir. In this memoir, we use an explicit description of the free operad functor in terms of trees with inputs, which we define in this section.

In §2.5.2, we give an explicit construction of a cosimplicial frame associated to the cobar construction $B^c(\mathcal{C})$ of a coaugmented non symmetric cooperad as a sequence.

In §2.5.3, we finally prove Theorem G, which gives a description of a mapping space $\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}),\mathcal{P})$ in the category of non symmetric operads as the simplicial Maurer-Cartan set associated to the brace algebra $\mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}},\overline{\mathcal{P}})$. This gives a computation of the connected components and the homotopy groups of $\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}),\mathcal{P})$ by using Theorem E.

2.5.1 The free operad functor and the model structure on $\mathcal{O}p$

We first recall the definition of the free operad functor and the model structure on operads. We will mostly follow conventions of [Mur11]. Let $\mathrm{Seq}_{\mathbb{K}}$ be the category of sequences in $\mathrm{dgMod}_{\mathbb{K}}$. Recall that we have an obvious model structure on $\mathrm{Seq}_{\mathbb{K}}$ which is defined arity wise, using the standard model structure on $\mathrm{dgMod}_{\mathbb{K}}$.

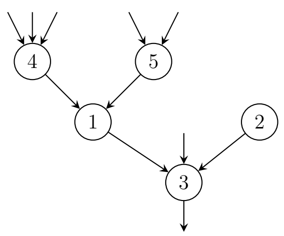
The model structure on the category of non symmetric operads $\mathcal{O}p$ is obtained by transferring the model structure of $\mathrm{Seq}_{\mathbb{K}}$ from an adjunction

$$\mathcal{F} : \mathrm{Seq}_{\mathbb{K}} \rightleftarrows \mathcal{O}p : \omega \ ,$$

where $\omega : \mathcal{O}p \longrightarrow \mathrm{Seq}_{\mathbb{K}}$ is the functor which forgets the operad structure. The left adjoint $\mathcal{F} : \mathrm{Seq}_{\mathbb{K}} \longrightarrow \mathcal{O}p$ is the *free operad* functor, for which we recall the construction.

We define the notion of tree with inputs, which is analogue to the notion of "planted planar tree with inputs" given in [Mur11, Definition 3.4].

Definition 2.5.1. *Let $n \geq 0$. A (planar)* tree with inputs *is the data of a tree $T \in \mathcal{PRT}(n)$ and, for each vertex of T, an integer which represents the number of ingoing arrows, which may includes some edges of T. We also add an outgoing arrow on the root of T. The ingoing arrows with only one vertex of T are called the* inputs *of the tree.*



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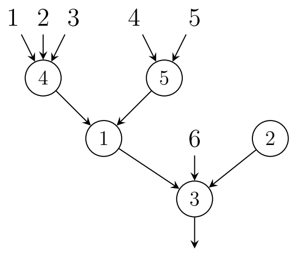
We usually denote by $\underline{T}$ any tree with inputs with underlying tree $T \in \mathcal{PRT}$. We call T the shape *of $\underline{T}$, and set* $\mathrm{Shape}(\underline{T}) = T$. *We also set $V_{\underline{T}} = V_T$. For every vertex $v \in V_{\underline{T}}$, we denote by $\mathrm{val}_{\underline{T}}(v)$ the number of ingoing arrows which go to v. We denote by $\underline{\mathcal{PRT}_k}(n)$ the set of trees with n vertices and k inputs and $\underline{\mathcal{T}ree_k}(n) = \mathbb{K}[\underline{\mathcal{PRT}_k}(n)]$.*

As in Definition 2.1.20, we can consider trees with inputs $\underline{T} \in \underline{\mathcal{PRT}_k}(a_1 < \cdots < a_n)$ in a general totally ordered finite set $a_1 < \cdots < a_n$. We say that $\underline{T}$ is *canonical* (or *in the canonical order*) if its shape $\mathrm{Shape}(\underline{T}) \in \mathcal{PRT}(a_1 < \cdots < a_n)$ is canonical.

For every tree $\underline{T} \in \underline{\mathcal{PRT}_k}(n)$, we endow the inputs with the canonical labeling from 1 to k obtained by following the canonical order of T. For instance, the tree given in the definition is seen as



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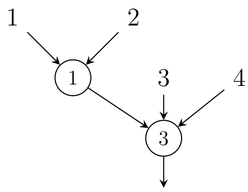
As for trees in $\mathcal{PRT}$, we have the following definitions.

Definition 2.5.2. *Let $\underline{T}$ be a tree with inputs and with underlying shape $T \in \mathcal{PRT}$.*

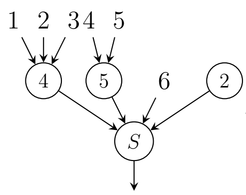
— *A* subtree *of $\underline{T}$ is the data of a subtree S of T, endowed with the unique choice of arrows such that, for every $v \in V_{\underline{S}} \subset V_{\underline{T}}$, we have* $\mathrm{val}_{\underline{S}}(v) = \mathrm{val}_{\underline{T}}(v)$.

— *If $\underline{S}$ is a subtree of $\underline{T}$, we denote by $\underline{T}/\underline{S}$ the tree of shape T/S obtained by contracting the tree $\underline{S}$ on the tree with only one vertex, denoted by S, with the same number of inputs as $\underline{S}$.*

For instance, if we consider the above tree with inputs $\underline{T}$, then the following tree with inputs

$\underline{S} =$ 

is a subtree of $\underline{T}$ such that

$\underline{T}/\underline{S} =$ 

.

Let $p, q, n, m \geq 0, 1 \leq i \leq p$ and $\underline{U} \in \underline{\mathcal{T}ree_p}(n), \underline{V} \in \underline{\mathcal{T}ree_q}(m)$. We let $\underline{U} \circ_i \underline{V}$ to be the tree in $\underline{\mathcal{T}ree_{p+q-1}}(n+m)$ given by the attachment of the unique outgoing arrow of $\underline{V}$ to the i-th ingoing arrow of $\underline{U}$. This defines a morphism

$$\circ_i : \underline{\mathcal{T}ree_p}(n) \otimes \underline{\mathcal{T}ree_q}(m) \longrightarrow \underline{\mathcal{T}ree_{p+q-1}}(n+m).$$

Lemma 2.5.3. *Let $\underline{\mathcal{T}ree}$ be the sequence defined by*

$$\underline{\mathcal{T}ree}(k) = \bigoplus_{n\geq 0} \underline{\mathcal{T}ree_k}(n).$$

Then the morphisms $\circ_i : \underline{\mathcal{T}ree_p} \otimes \underline{\mathcal{T}ree_q} \longrightarrow \underline{\mathcal{T}ree_{p+q-1}}$ endow the sequence $\underline{\mathcal{T}ree}$ with the structure of an operad.

Using this notion of tree, we set

$$\mathcal{F}(M)(k) = \bigoplus_{n\geq 0} \left(\bigoplus_{\underline{T}\in \underline{\mathcal{PRT}_k}(n)} \underline{T} \otimes \bigotimes_{i=1}^{n} M(\mathrm{val}_{\underline{T}}(i)) \right)_{\Sigma_n}$$

where, in the sum, we consider the action of Σ_n on $\underline{\mathcal{PRT}_k}(n)$ by the permutation of the labels of the vertices, and the action of Σ_n on $\bigotimes_{i=1}^{n} M(\mathrm{val}_{\underline{T}}(i))$ by permutations. The operadic structure of $\mathcal{F}(M)$ is given by the operadic structure of $\underline{\mathcal{T}ree}$ and the concatenation of the elements in M. We denote by $\mathcal{F}_{(\underline{T})}(M) = \bigotimes_{i=1}^{n} M(\mathrm{val}_{\underline{T}}(i))$ the $\underline{T}$-component of $\mathcal{F}(M)$ associated to some tree $\underline{T} \in \underline{\mathcal{PRT}_k}(n)$.

We can check that the functor $\mathcal{F} : \mathrm{Seq}_{\mathbb{K}} \longrightarrow \mathcal{O}p$ is left adjoint to the forgetful functor $\omega : \mathcal{O}p \longrightarrow \mathrm{Seq}_{\mathbb{K}}$ which forgets the operad structure:

$$\mathcal{F} : \mathrm{Seq}_{\mathbb{K}} \rightleftarrows \mathcal{O}p : \omega \ .$$

This adjunction implies the following result.

Proposition 2.5.4 (see [Mur11, Theorem 1.1])**.** *The category $\mathcal{O}p$ is endowed with a cofibrantly generated model structure such that the forgetful functor $\omega : \mathcal{O}p \longrightarrow \mathrm{Seq}_{\mathbb{K}}$ creates weak-equivalences and fibrations. Cofibrations are given by the left lifting property with respect to acyclic fibrations.*

Remark 2.5.5. *In the following subsections, we also use the notion of cofree cooperad generated by a sequence M such that $M(0) = 0$. For every $k \geq 0$ and $n \geq 1$, let $\underline{\mathcal{PRT}_k}^0(n)$ be the subset of $\underline{\mathcal{PRT}_k}(n)$ given by trees $\underline{T}$ such that $\mathrm{val}_{\underline{T}}(v) \neq 0$ for every $v \in V_{\underline{T}}$. Let $\underline{\mathcal{T}ree_k}^0(n) := \mathbb{K}[\underline{\mathcal{PRT}_k}^0(n)]$. Then the sequence $\underline{\mathcal{T}ree}^0$ defined by*

$$\underline{\mathcal{T}ree}^0(n) = \bigoplus_{k\geq 0} \underline{\mathcal{T}ree_k}^0(n)$$

is a suboperad of $\underline{\mathcal{T}ree}$ such that, for every $n \geq 0$, the $\mathbb{K}$-module $\underline{\mathcal{T}ree}^0(n)$ is finite dimensional. By Remark 2.1.7, the dual symmetric sequence $(\underline{\mathcal{T}ree}^0)^\vee$ is endowed with the structure of a cooperad. We then define

$$\mathcal{F}^c(M)(k) = \bigoplus_{n\geq 1} \left(\bigoplus_{\underline{T}\in \underline{\mathcal{PRT}_k}^0(n)} \underline{T}^\vee \otimes \bigotimes_{i=1}^{n} M(\mathrm{val}_{\underline{T}}(i)) \right)^{\Sigma_n}$$

where we consider the action of Σ_n on $\underline{T}^\vee$ by permutation of the vertices, and the action of Σ_n on $\bigotimes_{i=1}^{n} M(\mathrm{val}_{\underline{T}}(i))$ by permutations. We endow $\mathcal{F}^c(M)$ with the cooperad *structure given by the cooperadic structure of* $(\underline{\mathcal{T}ree}^0)^\vee$*, and by the deconcatenation coproduct in the tensor coalgebra of* $\bigoplus_{n\geq 1} M(n)$*. As for operads, we have an adjunction*

$$\omega : \mathcal{O}p^c \rightleftarrows \mathrm{Seq}_{\mathbb{K}} : \mathcal{F}^c$$

where $\omega : \mathcal{O}p^c \longrightarrow \mathrm{Seq}_{\mathbb{K}}$ *is the functor which forgets the cooperad structure.*

We will need to consider operadic compositions (resp. cooperadic cocompositions) shaped on trees with inputs. This can be formalized as follows. Let $\mathcal{P}$ be an augmented operad $\mathcal{P} \simeq I \oplus \overline{\mathcal{P}}$ and $\mathcal{C}$ be a coaugmented cooperad $\mathcal{C} \simeq I \oplus \overline{\mathcal{C}}$ such that $\mathcal{P}(0) = \mathcal{C}(0) = 0$ and $\mathcal{P}(1) = \mathcal{C}(1) = \mathbb{K}$. By the universal property satisfied by $\mathcal{F}$, we have a unique operad morphism $\mathcal{F}(\mathcal{P}) \longrightarrow \mathcal{P}$ which reduces to the identity on $\mathcal{P} \subset \mathcal{F}(\mathcal{P})$. Analogously, we have a unique cooperad morphism $\mathcal{C} \longrightarrow \mathcal{F}^c(\mathcal{C})$ whose projection on $\mathcal{C}$ is given by the identity on $\mathcal{C}$.

Definition 2.5.6. *Let* $k \geq 1$ *and* $\underline{T} \in \underline{\mathcal{PRT}_k}^0$*. We define* $\gamma_{(\underline{T})} : \mathcal{F}_{(\underline{T})}(\overline{\mathcal{P}}) \longrightarrow \overline{\mathcal{P}}$ *and* $\Delta_{(\underline{T})} : \overline{\mathcal{C}} \longrightarrow \mathcal{F}^c_{(\underline{T})}(\overline{\mathcal{C}})$ *by the composites*

$$\gamma_{(\underline{T})} : \mathcal{F}_{(\underline{T})}(\overline{\mathcal{P}}) \hookrightarrow \mathcal{F}_{(\underline{T})}(\mathcal{P}) \xrightarrow{\gamma} \mathcal{P} \twoheadrightarrow \overline{\mathcal{P}} \ ;$$

$$\Delta_{(\underline{T})} : \overline{\mathcal{C}} \hookrightarrow \mathcal{C} \xrightarrow{\Delta} \mathcal{F}^c_{(\underline{T})}(\mathcal{C}) \twoheadrightarrow \mathcal{F}^c_{(\underline{T})}(\overline{\mathcal{C}}) \ .$$

For every $p, q, n, m \geq 0$ and $1 \leq i \leq p$, we define a morphism

$$\bullet_i : \underline{\mathcal{T}ree_p}(n) \otimes \underline{\mathcal{T}ree_q}(m) \longrightarrow \underline{\mathcal{T}ree_p}(n+m-1)$$

by the following. Let $\underline{U} \in \mathcal{PRT}_p(n)$ and $\underline{V} \in \mathcal{PRT}_q(m)$. If the number of arrows on the i-th vertex of $\underline{U}$ is not q, then $\underline{U} \bullet_i \underline{V} = 0$. Else, we define $\underline{U} \bullet_i \underline{V}$ as the unique tree obtained by putting $\underline{V}$ in the i-th vertex of $\underline{U}$, and attaching the ingoing arrows of the i-th vertex of $\underline{U}$ into the inputs of $\underline{V}$.

Lemma 2.5.7. *Let* $\underline{T}$ *be a tree with inputs and* $\underline{S}$ *be a subtree of* $\underline{T}$*. Then* $\underline{T}/\underline{S} \bullet_S \underline{S} = \underline{T}$*.*

Proof. It is an immediate consequence of the definitions. □

The morphisms defined in Definition 2.5.6 also behave well with the compositions $\circ_i$ and $\bullet_i$.

Lemma 2.5.8. *Let* $k \geq 1$*,* $\underline{T} \in \underline{\mathcal{PRT}_k}^0(n)$ *and* $\underline{S} \subset \underline{T}$*. Then*

$$\gamma_{(\underline{T}/\underline{S})} \circ_{\underline{S}} \gamma_{(\underline{S})} = \gamma_{(\underline{T})} \ ; \ \Delta_{(\underline{T}/\underline{S})} \circ_{\underline{S}} \Delta_{(\underline{S})} = \Delta_{(\underline{T})}$$

in the endomorphism operad $\mathrm{End}_{\bigoplus_{n\geq 2}\mathcal{P}(n)}$ *and in the coendomorphism operad* $\mathrm{CoEnd}_{\bigoplus_{n\geq 2}\mathcal{C}(n)}$ *respectively.*

Proof. These are direct consequences of the (co)associativity axioms in a (co)operad. □

2.5.2 A cosimplicial frame for $B^c(\mathcal{C})$

Let $\mathcal{C} \simeq I \oplus \overline{\mathcal{C}}$ be a coaugmented non-symmetric cooperad with $\mathcal{C}(0) = 0$ and $\mathcal{C}(1) = \mathbb{K}$. The goal of this subsection is to construct a cosimplicial frame $B^c(\mathcal{C}) \otimes \Delta^n$ associated to $B^c(\mathcal{C})$. We will explicitly define $B^c(\mathcal{C}) \otimes \Delta^n$ as the free operad induced by a cooperad up to homotopy that will be given by $\overline{\mathcal{C}} \otimes N_*(\Delta^n)$.

Let E be a $\mathcal{E}$-coalgebra. We endow the operad $\mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1}E)$ with a general twisting morphism such that if $E = N_*(\Delta^0) \simeq \mathbb{K}$, then $\mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1}E) \simeq B^c(\mathcal{C})$. Explicitly, we construct $\beta^E : \overline{\mathcal{C}} \otimes \Sigma^{-1}E \longrightarrow \mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1}E)$ such that the morphism $\partial_{\beta^E} : \mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1}E) \longrightarrow \mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1}E)$ obtained from β^E by the Leibniz rule is a twisting morphism. If we denote by d the differential induced by $\overline{\mathcal{C}} \otimes \Sigma^{-1}E$ on $\mathcal{F}^c(\overline{\mathcal{C}} \otimes \Sigma^{-1}E)$, then the morphism β^E shall needs (see [LV12] or [Fre09b] for instance) to be such that

$$d(\beta^E) + \partial_{\beta^E}\beta^E = 0.$$

Let $k \geq 1$ and $\underline{T} \in \underline{\mathcal{PRT}_k}^0$ be a canonical tree with inputs with shape $T \in \mathcal{PRT}$. We define $\beta^E_{(\underline{T})} : \overline{\mathcal{C}} \otimes \Sigma^{-1}E \longrightarrow \mathcal{F}_{(\underline{T})}(\overline{\mathcal{C}} \otimes \Sigma^{-1}E)$ by

$$\beta^E_{(\underline{T})} = \Delta_{(\underline{T})} \widetilde{\otimes} \Lambda\mu^E_T$$

where, for every $\mu \in \mathcal{E}(n)$, we denote by μ^E the morphism in $\mathrm{Hom}(E, E^{\otimes n})$ given by the $\mathcal{E}$-coalgebra structure E.

This gives a well defined morphism of sequences $\beta^E : \overline{\mathcal{C}} \otimes \Sigma^{-1}E \longrightarrow \mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1}E)$ by summing over all canonical trees T. Note that such a sum of morphisms is will defined on $\overline{\mathcal{C}} \otimes \Sigma^{-1}E$ since $\overline{\mathcal{C}}(0) = \overline{\mathcal{C}}(1) = 0$. It is also natural in E by definition.

Proposition 2.5.9. *The morphism β^E defined above satisfies*

$$d(\beta^E) + \partial_{\beta^E}\beta^E = 0.$$

We thus have a derivation of operads $d + \partial_{\beta^E}$ on $\mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1}E)$.

Proof. It is sufficient to prove the formula on $\overline{\mathcal{C}} \otimes \Sigma^{-1}E$. Let $\underline{T}$ be a tree with inputs with shape a canonical tree $T \in \mathcal{PRT}$. We show that the $\underline{T}$-component of the morphism $d(\beta^E) + \partial_{\beta^E}\beta^E$ is 0. First, we have

$$d(\beta^E_{(\underline{T})}) = \Delta_{(\underline{T})} \widetilde{\otimes} d(\Lambda\mu_T)^{\Sigma E},$$

since the cooperadic structure on $\mathcal{C}$ is preserved by its differential. Next, we have by definition

$$(\partial_{\beta^E}\beta^E)_{(\underline{T})} = \sum_{\underline{S} \subset \underline{T}} (\Delta_{(\underline{T}/\underline{S})} \circ_{\mathrm{Shape}(\underline{S})} \Delta_{(\underline{S})}) \widetilde{\otimes} (\Lambda\mu_{T/\mathrm{Shape}(\underline{S})} \circ_{\mathrm{Shape}(\underline{S})} \Lambda\mu_{\mathrm{Shape}(\underline{S})})^{\Sigma E}.$$

Note that taking a subtree $\underline{S}$ of $\underline{T}$ is equivalent to taking a subtree S of T. We thus have

$$(\partial_{\beta^E}\beta^E)_{(\underline{T})} = \sum_{S \subset T} \Delta_{(\underline{T})} \widetilde{\otimes} (\Lambda\mu_{T/S} \circ_S \Lambda\mu_S)^{\Sigma E}.$$

The proposition follows by Theorem 2.3.3. $\square$

We can now construct a cosimplicial frame for $B^c(\mathcal{C})$. Recall that the normalized chain complex $N_*(\Delta^n)$ admits a structure of a $\mathcal{E}$-coalgebra.

Definition 2.5.10. *Let $n \geq 0$. We set*

$$B^c(\mathcal{C}) \otimes \Delta^n = (\mathcal{F}(\overline{\mathcal{C}} \otimes \Sigma^{-1} N_*(\Delta^n)), \partial_{\beta^{N_*(\Delta^n)}}).$$

We immediately see that $B^c(\mathcal{C}) \otimes \Delta^\bullet$ defines a cosimplicial object in the category of non symmetric operads. By Corollary 2.3.9, we also have that $B^c(\mathcal{C}) \otimes \Delta^0 = B^c(\mathcal{C})$.

Recall from [Fre17b, §3.2.2-§3.2.3] that a *cosimplicial frame* associated to $B^c(\mathcal{C})$ is a cosimplicial set $B^c(\mathcal{C}) \otimes \Delta^\bullet$ such that, for every $n \geq 0$, the morphism $B^c(\mathcal{C}) \otimes \Delta^n \longrightarrow B^c(\mathcal{C}) \otimes \Delta^0$ is a weak equivalence and the morphism $B^c(\mathcal{C}) \otimes \partial\Delta^n \longrightarrow B^c(\mathcal{C}) \otimes \Delta^n$ is a cofibration for every $n \geq 0$.

Theorem 2.5.11. *The cosimplicial object $B^c(\mathcal{C}) \otimes \Delta^\bullet$ defines a cosimplicial frame for $B^c(\mathcal{C})$ in the category of operads.*

Proof. Since the morphisms $N_*(\partial\Delta^n) \longrightarrow N_*(\Delta^n)$ are cofibrations for every $n \geq 0$, the morphisms $\overline{\mathcal{C}} \otimes \Sigma^{-1} N_*(\partial\Delta^n) \longrightarrow \overline{\mathcal{C}} \otimes \Sigma^{-1} N_*(\Delta^n)$ are cofibrations (see [Fre09b, Proposition 1.4.13]). We now prove that $B^c(\mathcal{C}) \otimes \Delta^n \longrightarrow B^c(\mathcal{C})$ is a weak equivalence. We first note that $B^c(\mathcal{C}) \otimes \Delta^n$ admits a natural filtration $(F_p(B^c(\mathcal{C}) \otimes \Delta^n))_p$ defined by

$$F_p(B^c(\mathcal{C}) \otimes \Delta^n) = \bigoplus_{k \geq 1} \bigoplus_{\substack{\underline{T} \in \underline{\mathcal{PRT}_k}^0 \\ \underline{T} \text{ canonical} \\ |\underline{T}| \geq p+1}} \underline{T} \otimes \mathcal{F}_{(\underline{T})}(\overline{\mathcal{C}} \otimes \Sigma^{-1} N_*(\Delta^n)).$$

By definition, the differential ∂^n preserves this filtration. We thus have a spectral sequence which is convergent dimension-wise:

$$E^0_q \Rightarrow H_*(B^c(\mathcal{C}) \otimes \Delta^n)$$

where we have set

$$E^0_q = \bigoplus_{k \geq 1} \bigoplus_{\substack{\underline{T} \in \underline{\mathcal{PRT}_k}^0 \\ \underline{T} \text{ canonical} \\ |\underline{T}| = q+1}} \underline{T} \otimes \mathcal{F}_{(\underline{T})}(\overline{\mathcal{C}} \otimes \Sigma^{-1} N_*(\Delta^n)).$$

Because the twisting part $\partial^{N_*(\Delta^n)}$ increases the number of vertices, the differential is reduced to the internal differential d on E^0_q. Because $\overline{\mathcal{C}} \otimes N_*(\Delta^n) \longrightarrow \overline{\mathcal{C}}$ is a weak equivalence, we have that the morphism $N_*(\Delta^n) \longrightarrow N_*(\Delta^0)$ induces a weak equivalence on E^0_p for all p. It then induces a weak equivalence from $B^c(\mathcal{C}) \otimes \Delta^n$ to $B^c(\mathcal{C})$.

We then have the result. $\square$

2.5.3 Computation of $\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}),\mathcal{P})$

In this last subsection, we give an explicit description of a mapping space $\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}),\mathcal{P})$. We know from Theorem 2.5.11 that we can set

$$\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}),\mathcal{P})_n = \mathrm{Mor}_{\mathcal{O}p}(B^c(\mathcal{C})\otimes\Delta^n,\mathcal{P})$$

for every $n\geq 0$. The goal of this subsection is to link this object with some $\Gamma\widehat{\Lambda\mathcal{PL}}_\infty$-algebra structure on $\mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}\otimes\Sigma^{-1}N_*(\Delta^n),\overline{\mathcal{P}})$. We consider the dg $\mathbb{K}$-module

$$\mathcal{L}(\mathrm{Hom}(\overline{\mathcal{C}}\otimes N_*(\Delta^n),\overline{\mathcal{P}})) = \bigoplus_{k\geq 2}\mathrm{Hom}(\overline{\mathcal{C}}(k)\otimes N_*(\Delta^n),\overline{\mathcal{P}}(k)).$$

This dg $\mathbb{K}$-module is endowed with a filtration defined by

$$F_p(\mathcal{L}(\mathrm{Hom}(\overline{\mathcal{C}}\otimes N_*(\Delta^n),\overline{\mathcal{P}}))) = \bigoplus_{k\geq p+1}\mathrm{Hom}(\overline{\mathcal{C}}(k)\otimes N_*(\Delta^n),\overline{\mathcal{P}}(k)),$$

so that $\mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}\otimes N_*(\Delta^n),\overline{\mathcal{P}})$ is the completion of $\mathcal{L}(\mathrm{Hom}(\overline{\mathcal{C}}\otimes N_*(\Delta^n),\overline{\mathcal{P}}))$ with respect to this filtration.

Lemma 2.5.12. *The dg $\mathbb{K}$-module $\mathcal{L}(\mathrm{Hom}(\overline{\mathcal{C}}\otimes N_*(\Delta^n),\overline{\mathcal{P}}))$ is endowed with the structure of a $\mathcal{B}race\underset{\mathrm{H}}{\otimes}\mathcal{E}$-algebra defined by*

$$(T\otimes\mu)(f_1,\ldots,f_k) = \pm\sum_{\underline{T}\in\mathrm{Shape}^{-1}(T)}\gamma_{(\underline{T})}\circ(f_1\otimes\cdots\otimes f_k)\circ(\Delta_{(\underline{T})}\widetilde{\otimes}\mu^{N_*(\Delta^n)}),$$

for any $T\in\mathcal{PRT}(k)$, $\mu\in\mathcal{E}(k)$ and $f_1,\ldots,f_k\in\mathcal{L}(\mathrm{Hom}(\overline{\mathcal{C}}\otimes N_*(\Delta^n),\overline{\mathcal{P}}))$ homogeneous, where we consider the tensor product $\widetilde{\otimes}$ (see Definition 2.1.1). The sign is yielded by the commutation of μ with the f_i's. Note that the sum is finite point-wise since we have supposed that $\mathcal{C}(0)=0$.

Proof. Let $\sigma\in\Sigma_n$. By definition of $\gamma_{(\underline{T})}$ and $\Delta_{(\underline{T})}$ for every $\underline{T}\in\underline{\mathcal{PRT}}$, we have

$$(\sigma\cdot T\otimes\sigma\cdot\mu)(f_1,\ldots,f_k) = \pm(T\otimes\mu)(f_{\sigma^{-1}(1)},\ldots,f_{\sigma^{-1}(k)}),$$

where we consider the action of σ on $(\overline{\mathcal{C}}\otimes N_*(\Delta^n))^{\otimes k}$ by permutation of the tensors.

We now prove the compatibility with the operadic structure. Let $p,q\geq 0$ and $U\in\mathcal{PRT}(p), V\in\mathcal{PRT}(q),\mu\in\mathcal{E}(p),\nu\in\mathcal{E}(q)$ and $1\leq i\leq p$. By Lemma 2.5.8, we have

$$\begin{aligned}(U\otimes\mu)&(f_1,\ldots,f_{i-1},(V\otimes\nu)(f_i,\ldots,f_{i+q-1}),f_{i+q},\ldots,f_{p+q-1})\\ &= \pm\sum_{\substack{\underline{U}\in\mathrm{Shape}^{-1}(U)\\ \underline{V}\in\mathrm{Shape}^{-1}(V)}}\gamma_{(\underline{U}\bullet_i\underline{V})}\circ(f_1\otimes\cdots\otimes f_{p+q-1})\circ(\Delta_{(\underline{U}\bullet_i\underline{V})}\widetilde{\otimes}(\mu\circ_i\nu)^{N_*(\Delta^n)}).\end{aligned}$$

Now, write $U\circ_i V = T_1+\cdots+T_m$ for some $T_1,\ldots,T_m\in\mathcal{PRT}$. By definition of the composition product in $\mathcal{B}race$, for every $\underline{U}\in\mathrm{Shape}^{-1}(U)$ and $\underline{V}\in\mathrm{Shape}^{-1}(V)$, the tree $\underline{U} \bullet_i \underline{V}$ has shape T_j for some $1 \leq j \leq m$ given by the particular choice of attachments forced by $\underline{V}$. In the converse direction, for every tree $\underline{T}$ with shape T_j, there exists a unique subtree $\underline{V} \subset \underline{T}$ with shape V such that $\underline{U} := \underline{T}/\underline{V}$ has shape U. This then proves that

$$(U \otimes \mu)(f_1, \ldots, f_{i-1}, (V \otimes \nu)(f_i, \ldots, f_{i+q-1}), f_{i+q}, \ldots, f_{p+q-1}) \\ = \pm(U \circ_i V \otimes \mu \circ_i \nu)(f_1 \otimes \cdots \otimes f_{p+q-1}).$$

□

Proposition 2.5.13. *The dg $\mathbb{K}$-module $\mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}} \otimes \Sigma^{-1} N_*(\Delta^n), \overline{\mathcal{P}})$ is endowed with the structure of a $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebra.*

Proof. By Lemma 2.5.12 and Theorem 2.3.11, the dg $\mathbb{K}$-module $\Sigma\mathcal{L}(\mathrm{Hom}(\overline{\mathcal{C}}\otimes N_*(\Delta^n), \overline{\mathcal{P}}))$ is endowed with the structure of a $\Gamma\Lambda\mathcal{PL}_\infty$-algebra. By taking the completion, we obtain that $\mathrm{Hom}(\overline{\mathcal{C}} \otimes \Sigma^{-1} N_*(\Delta^n), \overline{\mathcal{P}})$ is a $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebra. □

From the definition of the $\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$-algebra structure on $\mathrm{Hom}(\overline{\mathcal{C}} \otimes N_*(\Delta^n), \overline{\mathcal{P}})$, we deduce a first computation of $\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}), \mathcal{P})$.

Corollary 2.5.14. *We have the isomorphism of simplicial sets*

$$\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}), \mathcal{P}) \simeq \mathcal{MC}(\mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}} \otimes \Sigma^{-1} N_*(\Delta^\bullet), \overline{\mathcal{P}})).$$

Our goal is now to link this computation with the simplicial Maurer-Cartan set of $\mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})$. Recall that $N_*(\Delta^n)$ has a basis given by increasing sequence of integers $0 \leq a_0 < \cdots < a_r \leq n$ which we denote by $\underline{a_0 \cdots a_r}$. We let $\mathcal{B}_n$ to be this basis.

Lemma 2.5.15. *Let $n \geq 0$. We set*

$$\begin{array}{rccc} \phi_n : & \mathcal{L}(\mathrm{Hom}(\overline{\mathcal{C}} \otimes N_*(\Delta^n), \overline{\mathcal{P}})) & \longrightarrow & \mathcal{L}(\mathrm{Hom}(\overline{\mathcal{C}}, \overline{\mathcal{P}})) \otimes N^*(\Delta^n) \\ & f & \longmapsto & \sum_{\underline{x} \in \mathcal{B}_n} f^{\underline{x}} \otimes \underline{x}^\vee \end{array}$$

where, for every $\underline{x} \in N_*(\Delta^n)$ and $f \in \mathcal{L}(\mathrm{Hom}(\overline{\mathcal{C}} \otimes N_*(\Delta^n), \overline{\mathcal{P}}))$, we denote by $f^{\underline{x}} \in \mathcal{L}(\mathrm{Hom}(\overline{\mathcal{C}}, \overline{\mathcal{P}}))$ the map defined by $f^{\underline{x}}(c) = (-1)^{|c||\underline{x}|} f(c \otimes \underline{x})$ for every $c \in \overline{\mathcal{C}}$.
Then ϕ_n is an isomorphism of $\mathcal{B}race \underset{\mathrm{H}}{\otimes} \mathcal{E}$-algebras.
Moreover, the sequence of isomorphisms $(\phi_n)_{n \geq 0}$ is compatible with the simplicial structures.

Proof. We first prove that ϕ_n commutes with the differentials. Let $f \in \mathcal{L}(\mathrm{Hom}(\overline{\mathcal{C}} \otimes N_*(\Delta^n), \overline{\mathcal{P}}))$. Then

$$d(\phi_n(f)) = \sum_{\underline{x} \in \mathcal{B}_n} d(f^{\underline{x}}) \otimes \underline{x}^\vee + \sum_{\underline{x} \in \mathcal{B}_n} (-1)^{|f|+|\underline{x}|} f^{\underline{x}} \otimes d(\underline{x}^\vee).$$

For every $c \in \overline{\mathcal{C}}$, we have that

$$d(f^{\underline{x}})(c) = (-1)^{|c||\underline{x}|} d(f(c \otimes \underline{x})) - (-1)^{|f|+|\underline{x}|(|c|+1)} f(d(c) \otimes \underline{x})$$

which gives

$$d(f^{\underline{x}}) = d(f)^{\underline{x}} + (-1)^{|f|} f^{d(\underline{x})}.$$

We thus obtain

$$d(\phi_n(f)) = \sum_{\underline{x}\in\mathcal{B}_n} d(f)^{\underline{x}} \otimes \underline{x}^\vee + \sum_{\underline{x}\in\mathcal{B}_n} (-1)^{|f|} \left(f^{d(\underline{x})} \otimes \underline{x}^\vee + (-1)^{|\underline{x}|} f^{\underline{x}} \otimes d(\underline{x}^\vee) \right).$$

It remains to prove that

$$\sum_{\underline{x}\in\mathcal{B}_n} f^{d(\underline{x})} \otimes \underline{x}^\vee = -\sum_{\underline{x}\in\mathcal{B}_n} (-1)^{|\underline{x}|} f^{\underline{x}} \otimes d(\underline{x}^\vee).$$

For every $\underline{x} \in \mathcal{B}_n$, we write $d(\underline{x}) = \sum_{\underline{x}\in\mathcal{B}_n} \lambda_{\underline{x}}^{\underline{y}} \underline{y}$ where $\lambda_{\underline{y}}^{\underline{x}} \in \{-1; 0; 1\}$. We thus have, for every $\underline{y} \in \mathcal{B}_n$,

$$d(\underline{y}^\vee) = -(-1)^{|\underline{y}|} \sum_{\underline{x}\in\mathcal{B}_n} \lambda_{\underline{x}}^{\underline{y}} \underline{x}^\vee.$$

We thus have

$$\begin{aligned} \sum_{\underline{x}\in\mathcal{B}_n} f^{d(\underline{x})} \otimes \underline{x}^\vee &= \sum_{\underline{x},\underline{y}\in\mathcal{B}_n} \lambda_{\underline{x}}^{\underline{y}} f^{\underline{y}} \otimes \underline{x}^\vee \\ &= -\sum_{\underline{y}\in\mathcal{B}_n} (-1)^{|\underline{y}|} f^{\underline{y}} \otimes d(\underline{y}^\vee). \end{aligned}$$

At the end, we have obtained that

$$d(\phi_n(f)) = \sum_{\underline{x}\in\mathcal{B}_n} d(f)^{\underline{x}} \otimes \underline{x}^\vee = \phi_n(d(f))$$

so that ϕ_n commutes with the differentials.

Now, let $f_1, \ldots, f_r \in \mathcal{L}(\mathrm{Hom}(\overline{\mathcal{C}} \otimes N_*(\Delta^n), \overline{\mathcal{P}}))$ be homogeneous elements. Let $T \in \mathcal{PRT}$ be a canonical tree with r vertices and $\mu \in \mathcal{E}(r)$. We have

$$\begin{aligned} &(T \otimes \mu)(\phi_n(f_1), \ldots, \phi_n(f_r)) \\ &= \sum_{\underline{T}\in\mathrm{Shape}^{-1}(T)} \sum_{\underline{x_1},\ldots,\underline{x_r}\in\mathcal{B}_n} \pm(\gamma_{(\underline{T})} \circ (f_1^{\underline{x_1}} \otimes \cdots \otimes f_r^{\underline{x_r}}) \circ \Delta_{(\underline{T})}) \otimes \mu^{N^*(\Delta^n)}(\underline{x_1}^\vee, \ldots, \underline{x_r}^\vee), \end{aligned}$$

where the sign is given by

$$\prod_{i<j} (-1)^{|\underline{x_i}|(|f_j|+|\underline{x_j}|)} \times \prod_{j=1}^{r} (-1)^{|\mu|(|f_j|+|\underline{x_j}|)}.$$

Now, for every $\underline{x} \in \mathcal{B}_n$, we write

$$\mu^{N_*(\Delta^n)}(\underline{x}) = \sum_{\underline{x_1},\ldots,\underline{x_r}\in\mathcal{B}_n} \lambda_{\underline{x}}^{\underline{x_1},\ldots,\underline{x_r}} \underline{x_1} \otimes \cdots \otimes \underline{x_r}$$

where $\lambda_{\underline{x}}^{\underline{x_1},\ldots,\underline{x_r}} \in \{-1; 0; 1\}$ by definition of the interval cuts operations. This gives, for every $\underline{x_1}, \ldots, \underline{x_r} \in \mathcal{B}_n$,

$$\mu^{N^*(\Delta^n)}(\underline{x_1}^\vee, \ldots, \underline{x_r}^\vee) = \sum_{\underline{x}\in\mathcal{B}_n} \pm \underline{x}^\vee$$

where the sign is given by

$$\lambda_{\underline{x}}^{\underline{x_1},\dots,\underline{x_r}} \prod_{i<j} (-1)^{|\underline{x_i}||\underline{x_j}|} \prod_{j=1}^{r} (-1)^{|\mu||\underline{x_j}|}.$$

We thus have

$$(T\otimes\mu)(\phi_n(f_1),\dots,\phi_n(f_r)) = \sum_{\underline{T}\in \mathrm{Shape}^{-1}(T)} \sum_{\underline{x},\underline{x_1},\dots,\underline{x_r}\in\mathcal{B}_n} \pm(\gamma_{(\underline{T})}\circ(f_1^{\underline{x_1}}\otimes\cdots\otimes f_r^{\underline{x_r}})\circ\Delta_{(\underline{T})})\otimes\underline{x}^\vee$$

where the sign is

$$\lambda_{\underline{x}}^{\underline{x_1},\dots,\underline{x_r}} \prod_{i<j} (-1)^{|\underline{x_i}||\underline{f_j}|} \prod_{j=1}^{r} (-1)^{|\mu||\underline{f_j}|}.$$

We now use that, for every $c\in\overline{\mathcal{C}}$,

$$(f_1^{\underline{x_1}}\otimes\cdots\otimes f_r^{\underline{x_r}})(\Delta_{(\underline{T})}(c)) = \pm(f_1\otimes\cdots\otimes f_r)\circ(\Delta_{(\underline{T})}(c)\widetilde{\otimes}(\underline{x_1}\otimes\cdots\otimes\underline{x_r}))$$

where the sign is given by

$$\prod_{i<j} (-1)^{|\underline{x_i}||f_j|} \times \prod_{j=1}^{r} (-1)^{|c||\underline{x_j}|}.$$

We deduce

$$(T\otimes\mu)(\phi_n(f_1),\dots,\phi_n(f_r))(c) = \sum_{\underline{T}\in \mathrm{Shape}^{-1}(T)} \sum_{\underline{x}\in\mathcal{B}_n} \pm\gamma_{(\underline{T})}((f_1\otimes\cdots\otimes f_r)(\Delta_{(\underline{T})}(c)\widetilde{\otimes}\mu^{N_*(\Delta^n)}(\underline{x})))$$

where the sign is

$$\prod_{j=1}^{r} (-1)^{|\mu||f_j|} \times (-1)^{|c|(|\underline{x}|+|\mu|)},$$

since, for every $\underline{x},\underline{x_1},\dots,\underline{x_r}\in\mathcal{B}_n$ such that $\lambda_{\underline{x}}^{\underline{x_1},\dots,\underline{x_r}}\neq 0$, we have $|\mu|-\sum_{i=1}^r|\underline{x_i}| = -|\underline{x}|$. We finally obtain

$$(T\otimes\mu)(\phi_n(f_1),\dots,\phi_n(f_r)) = \sum_{\underline{x}\in\mathcal{B}_n} ((T\otimes\mu)(f_1,\dots,f_r))^{\underline{x}}\otimes\underline{x}^\vee = \phi_n((T\otimes\mu)(f_1,\dots,f_r)).$$

We thus have proved that $\phi_n:\mathcal{L}(\mathrm{Hom}(\overline{\mathcal{C}}\otimes N_*(\Delta^n),\overline{\mathcal{P}}))\longrightarrow\mathcal{L}(\mathrm{Hom}(\overline{\mathcal{C}},\overline{\mathcal{P}}))\otimes N^*(\Delta^n)$ is a morphism of complete $\mathcal{B}race\underset{\mathrm{H}}{\otimes}\mathcal{E}$-algebras, and ϕ is obviously a bijection, with as inverse

$$\phi_n^{-1}(f\otimes\underline{x}^\vee) = (c\otimes\underline{x}\longmapsto(-1)^{|c||\underline{x}|}f(c))$$

for every $f\in\mathcal{L}(\mathrm{Hom}(\overline{\mathcal{C}},\overline{\mathcal{P}}))$ and $\underline{x}\in\mathcal{B}_n$.

The compatibility of the sequence $(\phi_n)_{n\geq 0}$ with the simplicial structures follows directly from the definition of the ϕ_n's. □

In particular, the map ϕ_n induces an isomorphism of $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty,-)$-algebras. We thus obtain Theorem G.

Theorem 2.5.16. *We have the identity*

$$\mathrm{Map}_{\mathcal{O}p}(B^c(\mathcal{C}), \mathcal{P}) = \mathcal{MC}_\bullet(\mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}})).$$

Proof. For every $n \geq 0$, since the morphism ϕ_n given in Lemma 2.5.15 preserves the filtrations, taking the completions gives an isomorphism

$$\mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}} \otimes \Sigma^{-1} N_*(\Delta^n), \overline{\mathcal{P}}) \xrightarrow{\simeq} \Sigma \mathrm{Hom}_{\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}, \overline{\mathcal{P}}) \otimes N^*(\Delta^n)$$

of $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebras for every $n \geq 0$. Since this isomorphism preserves the simplicial structures, we obtain the theorem. $\square$

2.6 A mapping space in the category of symmetric connected operads

In this last section, we show that we can describe a mapping space in the category of symmetric and connected operads as the degree-wise Maurer-Cartan set of some complete $\Gamma\Lambda\mathcal{PL}_\infty$-algebra.

In §2.6.1, we recall the construction of the free operad functor in the category of symmetric connected operads and the model structure on the latter category.

In §2.6.2, we use the surjection cooperad $\mathbf{Sur}_{\mathbb{K}}$ to obtain a Σ_*-cofibrant replacement $B^c(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}})$ of the cobar construction $B^c(\mathcal{C})$ associated to a symmetric cooperad $\mathcal{C}$ such that $\mathcal{C}(0) = 0$. We construct an explicit cosimplicial frame associated to $B^c(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}})$.

In §2.6.3, we finally deduce Theorem H which gives a computation of the mapping spaces in the category of symmetric connected operads in terms of a degree-wise simplicial Maurer-Cartan set of some $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebras.

2.6.1 The free symmetric operad functor and the model structure on $\Sigma\mathcal{O}p^0$

In this subsection, we recall the construction of the free operad functor in the category $\Sigma\mathcal{O}p$ and recall the model structure on the category of symmetric connected operads $\Sigma\mathcal{O}p^0$.

We have a functor $- \otimes \Sigma : \mathrm{Seq}_{\mathbb{K}} \longrightarrow \Sigma\mathrm{Seq}_{\mathbb{K}}$ defined, for every $M \in \mathrm{Seq}_{\mathbb{K}}$, by

$$(M \otimes \Sigma)(n) = M(n) \otimes \mathbb{K}[\Sigma_n],$$

where $M(n) \otimes \mathbb{K}[\Sigma_n]$ is endowed with the Σ_n action defined, for every $m \in M(n)$ and $\sigma, \tau \in \Sigma_n$, by

$$\sigma \cdot (m \otimes \tau) = m \otimes \sigma\tau.$$

The functor $-\otimes\Sigma$ fits in an adjunction

$$-\otimes\Sigma : \mathrm{Seq}_{\mathbb{K}} \rightleftarrows \Sigma\mathrm{Seq}_{\mathbb{K}} : \omega \ ,$$

where $\omega : \Sigma\mathrm{Seq}_{\mathbb{K}} \longrightarrow \mathrm{Seq}_{\mathbb{K}}$ is the functor which forgets the symmetric groups actions. We have the following result.

Proposition 2.6.1 (see [Fre09a, Proposition 11.4.A])**.** *The category $\Sigma\mathrm{Seq}_{\mathbb{K}}$ is endowed with a cofibrantly generated model category structure such that the forgetful functor $\omega : \Sigma\mathrm{Seq}_{\mathbb{K}} \longrightarrow \mathrm{Seq}_{\mathbb{K}}$ creates weak-equivalences and fibrations. Cofibrations are given by the left lifting property with respect to acyclic fibrations.*

In fact, the category $\Sigma\mathcal{O}p$ does not have a model structure but rather a *semi-model* structure (see [Spi01, Theorem 3]). We instead consider the subcategory $\Sigma\mathcal{O}p^0$ of operads $\mathcal{P}$ such that $\mathcal{P}(0) = 0$. Such an operad is said to be *connected.* We also denote by $\Sigma\mathrm{Seq}^0_{\mathbb{K}}$ the subcategory of $\Sigma\mathrm{Seq}_{\mathbb{K}}$ given by symmetric sequences M such that $M(0) = 0$. As for the non symmetric context, we are searching for a convenient adjunction

$$\mathcal{F} : \Sigma\mathrm{Seq}^0_{\mathbb{K}} \rightleftarrows \Sigma\mathcal{O}p^0 : \omega$$

where $\omega : \Sigma\mathcal{O}p^0 \longrightarrow \Sigma\mathrm{Seq}^0_{\mathbb{K}}$ is the functor which forgets the operad structure. To achieve this, recall that the functor $-\otimes\Sigma : \mathrm{Seq}_{\mathbb{K}} \longrightarrow \Sigma\mathrm{Seq}_{\mathbb{K}}$ restricts to a functor $-\otimes\Sigma : \mathcal{O}p \longrightarrow \Sigma\mathcal{O}p$ where, for every $\mathcal{P} \in \mathcal{O}p$, the operad structure on $\mathcal{P}\otimes\Sigma$ is defined by

$$\gamma((f\otimes\sigma)\otimes(g_1\otimes\tau_1)\otimes\cdots\otimes(g_n\otimes\tau_n)) = \pm\gamma(f\otimes g_{\sigma^{-1}(1)}\otimes\cdots\otimes g_{\sigma^{-1}(n)})\otimes\sigma(\tau_1,\ldots,\tau_n),$$

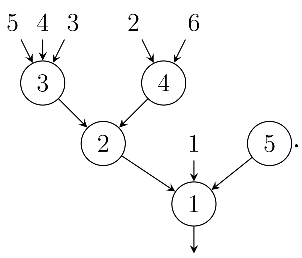
where we consider the composite of σ with $\tau_1,\ldots,\tau_n$ (see after Lemma 2.1.34).

Definition 2.6.2. *Consider the operad $\underline{\mathcal{T}ree}$ defined in Lemma 2.5.3. We set*

$$\Sigma\underline{\mathcal{T}ree} = \underline{\mathcal{T}ree}\otimes\Sigma.$$

For every $\underline{T} \in \Sigma\underline{\mathcal{T}ree}$, we denote by $V_{\underline{T}}$ the set of vertices, and by $\mathrm{val}_{\underline{T}}(v)$ the number of ingoing arrows on a vertex $v \in V_{\underline{T}}$.

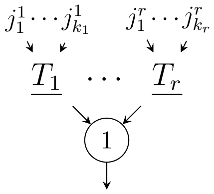
The elements of $\Sigma\underline{\mathcal{T}ree}(n)$ can then be seen as trees with inputs endowed with a choice of labeling on the inputs. We identify such a choice with a permutation in Σ_n. For instance, if we consider the tree with inputs $\underline{T} \in \underline{\mathcal{T}ree}$ given in Definition 2.5.1, then

$\underline{T}\otimes(643215) =$ 

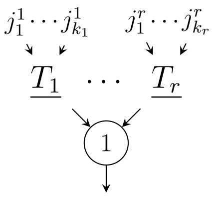
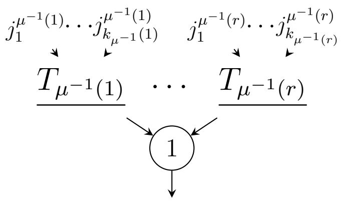
For every $M \in \Sigma\mathrm{Seq}_{\mathbb{K}}$, we consider the sequence

$$k \longmapsto \bigoplus_{n\geq 0} \bigoplus_{\underline{T}\in\underline{\mathcal{PRT}_k}(n)} \bigoplus_{\sigma\in\Sigma_k} (\underline{T}\otimes\sigma)\otimes\bigotimes_{i=1}^{n} M(\mathrm{val}_{\underline{T}}(i)).$$

This sequence is endowed with the structure of a non-symmetric operad given by the operadic structure of $\Sigma\underline{\mathcal{T}ree}$ and the concatenation of elements in the tensor algebra of $\bigoplus_{n\geq 0} M(n)$. In order to endow this sequence with the structure of a symmetric operad, we need to identify some elements. First, we endow this sequence with the Σ_n-action given by the left translation in Σ_n. For every $n \geq 0$, we identify the action of Σ_n on $\underline{T} \in \underline{\mathcal{PRT}_k}(n)$ given by the permutation of the vertices with the action of Σ_n on $\bigotimes_{i=1}^{n} M(\mathrm{val}_{\underline{T}}(i))$ given by the permutation of the factors. Next, consider a tree in $\Sigma\underline{\mathcal{T}ree}$ of the form

$\underline{T} :=$  ,

for some $r \geq 1$ and $\underline{T_1} \in \underline{\mathcal{PRT}_{k_1}}, \ldots, \underline{T_r} \in \underline{\mathcal{PRT}_{k_r}}$ with $k_1+\cdots+k_r = k$. Let $x \in M(r)$ and $Y \in \bigotimes_{i=2}^{n} M(\mathrm{val}_{\underline{T}}(i))$. For every $\mu \in \Sigma_r$, we make the identification

 $\otimes\, x \otimes Y \equiv$  $\otimes\, \mu\cdot x\otimes Y.$

We iterate such identifications by induction on the number of vertices of the tree, using the operadic structure. It is an immediate check that we obtain a symmetric operad, which we denote by $\mathcal{F}(M)$, such that the functor $\mathcal{F}$ fits in the left-right adjunction

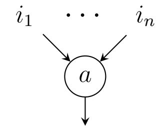
$$\mathcal{F} : \Sigma\mathrm{Seq}_{\mathbb{K}}^0 \rightleftarrows \Sigma\mathcal{O}p^0 : \omega\ .$$

Proposition 2.6.3 (see [Hin03, §3.3])**.** *The category $\Sigma\mathcal{O}p^0$ is endowed with a cofibrantly generated model structure such that the forgetful functor $\omega : \Sigma\mathcal{O}p^0 \longrightarrow \Sigma\mathrm{Seq}_{\mathbb{K}}^0$ creates weak-equivalences and fibrations. Cofibrations are given by the left lifting property with respect to acyclic fibrations.*

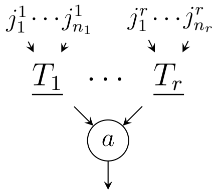
Let $M \in \Sigma\mathrm{Seq}_{\mathbb{K}}^0$. In the definition of $\mathcal{F}(M)$, we can restrict to trees in $\Sigma\underline{\mathcal{T}ree}$ such that every vertex has at least one input. We denote by $\Sigma\underline{\mathcal{T}ree}^0$ the underlying sequence. In the following sections, we use an explicit choice of set of representatives for trees in $\mathcal{F}(M)$. Such a choice can be made by taking *tree monomials* (see [DK10, §3.1]), for which we recall the definition.

Definition 2.6.4. *Let $\underline{T} \in \Sigma\underline{\mathcal{PRT}_{i_1<\cdots<i_n}}^0$ be a tree with m vertices. The tree $\underline{T}$ is a* tree monomial *if* $\mathrm{Shape}(\underline{T})$ *is in the canonical order, and if one of the three following conditions is fulfilled:*

— *$m = 0$ (so that $\underline{T}$ is the unit in the operad $\Sigma\underline{\mathcal{T}ree}$);*
— *$m = 1$ and $\underline{T}$ is of the form*

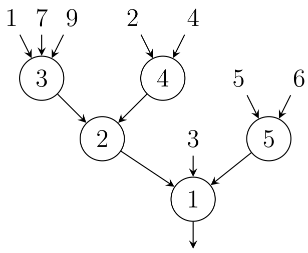
$\underline{T} =$ 

for some vertex a;
— *$m \geq 2$ and $\underline{T}$ is of the form*

$\underline{T} =$ 

for some vertex a and where $\underline{T_1} \in \Sigma\underline{\mathcal{PRT}_{\{j_1^1,\ldots,j_{n_1}^1\}}}^0 \ldots, \underline{T_r} \in \Sigma\underline{\mathcal{PRT}_{\{j_1^r,\ldots,j_{n_r}^r\}}}^0$ are tree monomials such that $\min(j_1^1, \ldots, j_{n_1}^1) < \cdots < \min(j_1^r, \ldots, j_{n_r}^r)$.

For instance, the tree



$\in \Sigma\underline{\mathcal{PRT}_{1<\cdots<9}}$

is a tree monomial. For every $n \geq 1$, we denote by $\mathcal{TM}_k(n)$ the set of tree monomials with n vertices and k inputs. We have the following result.

Proposition 2.6.5. *Let $M \in \Sigma\mathrm{Seq}_{\mathbb{K}}^0$. Then, for every $n \geq 1$,*

$$\mathcal{F}(M)(k) \simeq \bigoplus_{n\geq 0} \bigoplus_{\underline{T}\in\mathcal{TM}_k(n)} \underline{T} \otimes \bigotimes_{i=1}^{n} M(\mathrm{val}_{\underline{T}}(i)).$$

Proof. The proposition is obtained by iterating the second claim of Proposition 2.1.35, and by using the symmetry axioms in the operad $\mathcal{F}(M)$. □

As in the non-symmetric context, we have the following remark.

Remark 2.6.6. *Since, for every $k \geq 0$, the $\mathbb{K}$-module $\Sigma\underline{\mathcal{T}ree_k}^0$ is finite dimensional, we have that the dual symmetric sequence $(\Sigma\underline{\mathcal{T}ree}^0)^\vee$ is endowed with the structure of a cooperad. We can then define, for every $n \geq 0$,*

$$\mathcal{F}^c(M)(k) = \bigoplus_{n\geq 0} \bigoplus_{\underline{T}\in\mathcal{TM}_k(n)} \underline{T}^\vee \otimes \bigotimes_{i=1}^{n} M(\mathrm{val}_{\underline{T}}(i)).$$

One can show that this symmetric sequence is endowed with the structure of a cooperad given by the cooperad structure in $(\Sigma\underline{\mathcal{T}ree}^0)^\vee$*, and by the deconcatenation coproduct in the tensor algebra of* $\bigoplus_{n\geq 0} M(n)$*. As for the free operad functor, we have an adjunction*

$$\omega : (\Sigma\mathcal{O}p^c)^0 \rightleftarrows \Sigma\mathrm{Seq}^0_{\mathbb{K}} : \mathcal{F}^c$$

where we denote by $(\Sigma\mathcal{O}p^c)^0$ *the subcategory of* $\Sigma\mathcal{O}p^c$ *given by connected cooperads, and where* $\omega : (\Sigma\mathcal{O}p^c)^0 \longrightarrow \Sigma\mathrm{Seq}^0_{\mathbb{K}}$ *is the functor which forgets the cooperad structure.*

As for the non symmetric context, we consider operadic (resp. cooperadic) compositions (resp. cocompositions) shaped on trees with inputs. Let $\mathcal{P}$ be an augmented operad $\mathcal{P} \simeq I \oplus \overline{\mathcal{P}}$ and $\mathcal{C}$ be a coaugmented cooperad $\mathcal{C} \simeq I \oplus \overline{\mathcal{C}}$ such that $\mathcal{P}(0) = \mathcal{C}(0) = 0$ and $\mathcal{P}(1) = \mathcal{C}(1) = \mathbb{K}$. By the universal property satisfied by $\mathcal{F}$, we have a unique operad morphism $\mathcal{F}(\mathcal{P}) \longrightarrow \mathcal{P}$ which reduces to the identity on $\mathcal{P} \subset \mathcal{F}(\mathcal{P})$. Analogously, we have a unique cooperad morphism $\mathcal{C} \longrightarrow \mathcal{F}^c(\mathcal{C})$ whose composite with the projection on $\mathcal{C}$ is given by the identity on $\mathcal{C}$.

Definition 2.6.7. *Let* $\underline{T}$ *be a tree with inputs. We define* $\gamma_{(\underline{T})} : \mathcal{F}_{(\underline{T})}(\overline{\mathcal{P}}) \longrightarrow \overline{\mathcal{P}}$ *and* $\Delta_{(\underline{T})} : \overline{\mathcal{C}} \longrightarrow \mathcal{F}^c_{(\underline{T})}(\overline{\mathcal{C}})$ *by the composites*

$$\gamma_{(\underline{T})} : \mathcal{F}_{(\underline{T})}(\overline{\mathcal{P}}) \hookrightarrow \mathcal{F}_{(\underline{T})}(\mathcal{P}) \xrightarrow{\gamma} \mathcal{P} \twoheadrightarrow \overline{\mathcal{P}} \ ;$$

$$\Delta_{(\underline{T})} : \overline{\mathcal{C}} \hookrightarrow \mathcal{C} \xrightarrow{\Delta} \mathcal{F}^c_{(\underline{T})}(\mathcal{C}) \twoheadrightarrow \mathcal{F}^c_{(\underline{T})}(\overline{\mathcal{C}}) \ .$$

For every $p, q, n, m \geq 0$ and $1 \leq i \leq p$, as for the non symmetric context, we define a morphism

$$\bullet_i : \Sigma\underline{\mathcal{T}ree_p}(n) \otimes \Sigma\underline{\mathcal{T}ree_q}(m) \longrightarrow \Sigma\underline{\mathcal{T}ree_p}(n+m-1)$$

defined as follows. Let $\underline{U} \in \Sigma\underline{\mathcal{T}ree_p}(n)$ and $\underline{V} \in \Sigma\underline{\mathcal{T}ree_q}(m)$. If $\mathrm{val}_{\underline{U}}(i) \neq q$, we set $\underline{U} \bullet_i \underline{V} = 0$. Else, we define $\underline{U} \bullet_i \underline{V}$ as the tree obtained by changing the i-th vertex of $\underline{U}$ into the tree $\underline{V}$. The attachment of the q arrows on the i-th vertex of $\underline{U}$ on the tree $\underline{V}$ are given following the order of the labeling in $\underline{V}$.

As for the non symmetric context, we have the two following lemmas.

Lemma 2.6.8. *Let* $\underline{T}$ *be a tree with inputs and* $\underline{S}$ *be a subtree of* $\underline{T}$*. Then* $\underline{T}/\underline{S} \bullet_S \underline{S} = \underline{T}$*.*

Lemma 2.6.9. *Let* $\underline{T} \in \underline{\mathcal{PRT}}(n)$ *and* $\underline{S} \subset \underline{T}$*. Then*

$$\gamma_{(\underline{T}/\underline{S})} \circ_S \gamma_{(\underline{S})} = \gamma_{(\underline{T})} \ ; \ \Delta_{(\underline{T}/\underline{S})} \circ_S \Delta_{(\underline{S})} = \Delta_{(\underline{T})}$$

in the endomorphism operad $\mathrm{End}_{\bigoplus_{n\geq 2}\mathcal{P}(n)}$ *and in the coendomorphism operad* $\mathrm{CoEnd}_{\bigoplus_{n\geq 2}\mathcal{C}(n)}$ *respectively.*

2.6.2 A cosimplicial frame for $B^c(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}})$

Let $\mathbf{Sur}_{\mathbb{K}}$ be the surjection cooperad defined in [BCN23, Theorem A.1]. This cooperad is actually equal, as a symmetric sequence, to the surjection operad χ recalled in

§2.1.4. Note however that the cooperad structure on $\mathbf{Sur}_{\mathbb{K}}$ is not the cooperad structure obtained by dualizing the operad structure on χ. We have a weak-equivalence $B^c(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}}) \xrightarrow{\sim} B^c(\mathcal{C})$, which provides a Σ_*-cofibrant replacement of $B^c(\mathcal{C})$.

In this section, we construct an explicit cosimplicial frame associated to $B^c(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}})$ for every symmetric coaugmented cooperad $\mathcal{C}$. To be more precise, we construct a twisted differential $\partial^n : \mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1} N_*(\Delta^n)) \longrightarrow \mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1} N_*(\Delta^n))$ by an inductive process analogue to the one given in Theorem 2.3.7.

For every $k \geq 1$, we define $\Phi_n^0, H_n^0 : (\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1} N_*(\Delta^n))^{\otimes k} \longrightarrow (\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1} N_*(\Delta^n))^{\otimes k}$ by

$$\Phi_n^0 = id^{\otimes k}_{\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}}} \widetilde{\otimes} \phi_n^0;$$

$$H_n^0 = id^{\otimes k}_{\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}}} \widetilde{\otimes} h_n^0,$$

where we use the tensor product $\widetilde{\otimes}$ defined in Definition 2.1.1, and the morphisms $\phi_n^0, h_n^0 : (\Sigma^{-1} N_*(\Delta^n))^{\otimes k} \longrightarrow (\Sigma^{-1} N_*(\Delta^n))^{\otimes k}$ defined after Lemma 2.3.6. We extend Φ_n^0 and H_n^0 on $\mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1} N_*(\Delta^n))$ by using the identification given in Proposition 2.6.5. Note that the morphism H_n^0 does not preserve the action of the symmetric groups on $\mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1} N_*(\Delta^n))$.

Since the action of the symmetric groups on $\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}}$ is free, we can chose an explicit choice of representatives for the orbits. For every $n \geq 1$, we let $\mathbf{Sur}^{id}_{\mathbb{K}}(n)$ to be the dg $\mathbb{K}$-module generated by surjections $u \in \mathbf{Sur}_{\mathbb{K}}(n)$ of the form

$$u = \left| \begin{array}{cccc} u_0(1) & \cdots & u_0(r_0-1) & u_0(r_0) \\ \vdots & & \vdots & \\ u_{d-1}(1) & \cdots & u_{d-1}(r_{d-1}-1) & u_{d-1}(r_{d-1}) \\ u_d(1) & \cdots & u_d(r_d-1) & u_d(r_d) \end{array} \right.$$

with

$$u_0(1)\cdots u_0(r_0-1)\cdots u_{d-1}(1)\cdots u_{d-1}(r_{d-1}-1)\cdot u_d(1)\cdots u_d(r_d) = 1\cdots n.$$

We thus have an isomorphism of graded symmetric sequences $\mathbf{Sur}_{\mathbb{K}} \simeq \mathbf{Sur}^{id}_{\mathbb{K}} \otimes \Sigma$. This gives an isomorphism

$$\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}} \simeq (\mathcal{C} \otimes \mathbf{Sur}^{id}_{\mathbb{K}}) \otimes \Sigma$$

defined by sending $c \otimes (u \otimes \sigma) \in \mathcal{C} \underset{\mathrm{H}}{\otimes} (\mathbf{Sur}^{id}_{\mathbb{K}} \otimes \Sigma)$ to $(\sigma^{-1} \cdot c \otimes u) \otimes \sigma \in (\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}^{id}_{\mathbb{K}}) \otimes \Sigma$. Note that the differential $d_{\mathcal{C}}$ preserves such a decomposition, but not the differential $d_{\mathbf{Sur}_{\mathbb{K}}}$, since it does not preserve $\mathbf{Sur}^{id}_{\mathbb{K}}$. For every $l \geq 0$, we let $F_l \mathbf{Sur}^{id}_{\mathbb{K}}$ to be the sequence given by surjections of degree equal or less than l in $\mathbf{Sur}^{id}_{\mathbb{K}}$ and we set $F_l(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}^{id}_{\mathbb{K}}) = \mathcal{C} \underset{\mathrm{H}}{\otimes} F_l \mathbf{Sur}^{id}_{\mathbb{K}}$.

For every $n \geq 0$, we aim to define a derivation of operads on $\mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes$

$\Sigma^{-1}N_*(\Delta^n))$ which reduces to the internal differential of $\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1}N_*(\Delta^n)$ on trees with only one vertex. We denote by $d_{\mathcal{C}}, d_{\mathbf{Sur}_{\mathbb{K}}}$ and $d_{\Sigma^{-1}N_*(\Delta^n)}$ the corresponding differentials on $\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}_{\mathbb{K}}} \otimes \Sigma^{-1}N_*(\Delta^n)$. Let $\mathcal{C}_{ns} := \mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}}^{id} \otimes id$. We construct $\beta^n : \overline{\mathcal{C}_{ns}} \otimes \Sigma^{-1}N_*(\Delta^n) \longrightarrow \mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1}N_*(\Delta^n))$ which reduces to $d_{\mathbf{Sur}_{\mathbb{K}}} + d_{\Sigma^{-1}N_*(\Delta^n)}$ on trees with one vertex and which is such that

$$d_{\mathcal{C}}(\beta^n) + \partial^n\beta^n = 0,$$

where $\partial^n : \mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1}N_*(\Delta^n)) \longrightarrow \mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1}N_*(\Delta^n))$ is the morphism obtained from β^n by applying the Leibniz rule in $\mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1}N_*(\Delta^n))$.

In the following, we endow the sequence of operads $\mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1}N_*(\Delta^\bullet))$ with the structure of a cosimplicial set with as coface maps (resp. codegeneracy maps) the coface maps (resp. codegeneracy maps) of the cosimplicial set $\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1}N_*(\Delta^\bullet)$ taken tensor-wise. Recall that the cosimplicial relations are given by the following:

- If $i < j$, then $d^jd^i = d^id^{j-1}$;
- If $i < j$, then $s^jd^i = d^is^{j-1}$;
- $s^jd^j = s^jd^{j+1} = id$;
- If $i > j+1$, then $s^jd^i = d^{i-1}s^j$;
- If $i \leq j$, then $s^js^i = s^is^{j+1}$.

Note that we have an extra codegeneracy $s^{-1} : N_*(\Delta^n) \longrightarrow N_*(\Delta^{n-1})$ defined for every $0 \leq a_0 < \cdots < a_r \leq n$ by $s^{-1}(\underline{a_0 \cdots a_r}) = \underline{(a_0-1)\cdots(a_r-1)}$, with the convention $s^{-1}(\underline{a_0 \cdots a_r}) = 0$ if $a_0 = 0$. One can easily check that the above relations are still satisfied with the addition of this degeneracy.

Construction 2.6.10. *We define a sequence of degree -1 morphisms $\beta^n : \overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1}N_*(\Delta^n) \longrightarrow \mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}_{\mathbb{K}}} \otimes \Sigma^{-1}N_*(\Delta^n))$ by induction on $n \geq 0$. Let $\partial^n : \mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}_{\mathbb{K}}} \otimes \Sigma^{-1}N_*(\Delta^n)) \longrightarrow \mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}_{\mathbb{K}}} \otimes \otimes\Sigma^{-1}N_*(\Delta^n))$ be the morphism obtained from β^n by the Leibniz rule.*
We let β^0 to be such that $d_{\overline{\mathcal{C}}} + \partial^0$ is the differential of the cobar construction $B^c(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}}) \simeq \mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1}N_*(\Delta^0))$. We set $\beta^n_{(0)} = d_{\mathbf{Sur}_{\mathbb{K}}} + d_{\Sigma^{-1}N_*(\Delta^n)}$. For every $k \geq 1$, we define the component $\beta^n_{(k)}$ of β^n given by trees with $k+1$ vertices by induction on k and n. More precisely, we define $\beta^n_{(k)}$ on $F_l\overline{\mathcal{C}_{ns}} \otimes \Sigma^{-1}N_*(\Delta^n)$ by induction on $l \geq 0$. Let $c \in F_l\overline{\mathcal{C}_{ns}}$ and $\underline{x} \in N_*(\Delta^n)$ be a basis element.

- *If $\underline{x} \neq \underline{0\cdots n}$, let $0 \leq i \leq n$ be such that $\underline{x} = \underline{0\cdots(i-1)a_i\cdots a_r}$ with $i < a_i < \cdots < a_r \leq n$. We set*

$$\beta^n_{(k)}(c \otimes \Sigma^{-1}\underline{x}) = d^i\beta^{n-1}_{(k)}(c \otimes \Sigma^{-1}s^{i-1}\underline{x});$$

— *If $\underline{x} = \underline{0 \cdots n}$, we set*

$$\begin{aligned}\beta^n_{(k)}(c \otimes \Sigma^{-1}\underline{0 \cdots n}) &= (-1)^{|c|} H^0_n d^0 \beta^{n-1}_{(k)}(c \otimes \Sigma^{-1}\underline{0 \cdots (n-1)}) \\ &- H^0_n \beta^n_{(k)}(d_{\mathbf{Sur}_{\mathbb{K}}}(c) \otimes \Sigma^{-1}\underline{0 \cdots n}) - \sum_{\substack{p+q=k \\ p,q \neq 0}} H^0_n \partial^n_{(p)} \beta^n_{(q)}(c \otimes \Sigma^{-1}\underline{0 \cdots n}).\end{aligned}$$

The morphism $\beta^n_{(k)}$ is then extended on $F_l\overline{\mathcal{C}_{ns}} \otimes \Sigma \otimes \Sigma^{-1} N_*(\Delta^n)$ by symmetry.

Lemma 2.6.11. *For every $n, k \geq 0$, we have*

$$\forall 0 \leq j \leq n-1, d^j \beta^{n-1}_{(k)} = \beta^n_{(k)} d^j;$$

$$\forall 0 \leq j \leq n, s^j \beta^n_{(k)} = \beta^{n-1}_{(k)} s^j,$$

where we consider the morphisms $\beta^n_{(k)}$ defined in Construction 2.6.10.

Proof. Since the coface maps and codegeneracy maps preserve the action of the symmetric groups on $\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}_{\mathbb{K}}} \otimes \Sigma^{-1} N_*(\Delta^n)$, it is sufficient to prove it on $F_l\overline{\mathcal{C}_{ns}} \otimes \Sigma^{-1} N_*(\Delta^n)$ for every $l \geq 0$. Let $c \in \overline{\mathcal{C}_{ns}}$ and let $\underline{x} \in N_*(\Delta^n)$ be a basis element. We prove the formulas by induction on $n, k, l \geq 0$. The assertion is obviously true for $n = 0$, and for $n \geq 1$ and $k = 0$. We now suppose that $n, k \geq 1$.

We prove the first line of the lemma. Let $0 \leq j \leq n-1$. If $\underline{x} = \underline{0 \cdots (n-1)}$, then we indeed have $d^j \beta^{n-1}_{(k)}(c \otimes \Sigma^{-1}\underline{x}) = \beta^n_{(k)} d^j (c \otimes \Sigma^{-1}\underline{x})$ by definition of $\beta^n_{(k)}$. Suppose now that $\underline{x} \neq \underline{0 \cdots (n-1)}$. Then there exists $0 \leq i \leq n-1$ such that $\underline{x} = \underline{0 \cdots (i-1) a_i \cdots a_r}$ with $i < a_i < \cdots < a_r \leq n-1$.
If $j = i$, then

$$\begin{aligned}\beta^n_{(k)} d^j (c \otimes \Sigma^{-1}\underline{x}) &= \beta^n_{(k)}(c \otimes \Sigma^{-1} d^i \underline{x}) \\ &= d^i \beta^n_{(k)}(c \otimes \Sigma^{-1} s^{i-1} d^i \underline{x}) \\ &= d^j \beta^n_{(k)}(c \otimes \Sigma^{-1}\underline{x}),\end{aligned}$$

since $s^{i-1} d^i = id$. If $j > i$, then

$$\begin{aligned}\beta^n_{(k)} d^j (c \otimes \Sigma^{-1}\underline{x}) &= \beta^n_{(k)}(c \otimes \Sigma^{-1} d^j \underline{x}) \\ &= d^i \beta^{n-1}_{(k)}(c \otimes \Sigma^{-1} s^{i-1} d^j \underline{x}) \\ &= d^i \beta^{n-1}_{(k)}(c \otimes \Sigma^{-1} d^{j-1} s^{i-1} \underline{x}),\end{aligned}$$

since $s^{i-1} d^j = d^{j-1} s^{i-1}$. By induction hypothesis on $n-1$, we deduce

$$\begin{aligned}\beta^n_{(k)} d^j (c \otimes \Sigma^{-1}\underline{x}) &= d^i d^{j-1} \beta^{n-2}_{(k)}(c \otimes \Sigma^{-1} s^{i-1} \underline{x}) \\ &= d^j d^i \beta^{n-2}_{(k)}(c \otimes \Sigma^{-1} s^{i-1} \underline{x}) \\ &= d^j \beta^{n-1}_{(k)}(c \otimes \Sigma^{-1}\underline{x}).\end{aligned}$$

If $j < i$, then

$$\begin{aligned}\beta^n_{(k)} d^j (c \otimes \Sigma^{-1}\underline{x}) &= d^j \beta^{n-1}_{(k)}(c \otimes \Sigma^{-1} s^{j-1} d^j \underline{x}) \\ &= d^j \beta^{n-1}_{(k)}(c \otimes \Sigma^{-1}\underline{x}),\end{aligned}$$

since $s^{j-1}d^j = id$. We thus have proved that $d^j\beta^{n-1}_{(k)} = \beta^n_{(k)}d^j$.

We now prove the second line of the lemma. Let $\underline{x} \in N_*(\Delta^n)$. We first consider $\underline{x} = \underline{0\cdots n}$. Then, by definition of $\beta^n_{(k)}$,

$$\begin{aligned} s^j\beta^n_{(k)}(c\otimes\Sigma^{-1}\underline{0\cdots n}) &= (-1)^{|c|}s^jH^0_nd^0\beta^{n-1}_{(k)}(c\otimes\Sigma^{-1}\underline{0\cdots(n-1)}) \\ &- s^jH^0_n\beta^n_{(k)}(d_{\mathbf{Sur}_{\mathbb{K}}}(c)\otimes\Sigma^{-1}\underline{0\cdots n}) - \sum_{\substack{p+q=k\\ p,q\neq 0}} s^jH^0_n\partial^n_{(p)}\beta^n_{(q)}(c\otimes\Sigma^{-1}\underline{0\cdots n}). \end{aligned}$$

Since $s^jH^0_n = H^0_{n-1}s^j$, we have

$$\begin{aligned} s^j\beta^n_{(k)}(c\otimes\Sigma^{-1}\underline{0\cdots n}) &= (-1)^{|c|}H^0_{n-1}s^jd^0\beta^{n-1}_{(k)}(c\otimes\Sigma^{-1}\underline{0\cdots(n-1)}) \\ &- H^0_{n-1}s^j\beta^n_{(k)}(d_{\mathbf{Sur}_{\mathbb{K}}}(c)\otimes\Sigma^{-1}\underline{0\cdots n}) - \sum_{\substack{p+q=k\\ p,q\neq 0}} H^0_{n-1}s^j\partial^n_{(p)}\beta^n_{(q)}(c\otimes\Sigma^{-1}\underline{0\cdots n}). \end{aligned}$$

By induction hypothesis, we have that s^j commutes with $\partial^n_{(p)}\beta^n_{(q)}$ for every $p,q\neq 0$ such that $p+q=n$. Since $s^j(\underline{0\cdots n}) = 0$, we have that the sum in the above identity is 0. Analogously, by induction hypothesis on $l\geq 0$, we have that $s^j\beta^n_{(k)}(d_{\mathbf{Sur}_{\mathbb{K}}}(c)\otimes\Sigma^{-1}\underline{0\cdots n}) = 0$. If $j>0$, we have $s^jd^0 = d^0s^{j-1}$ so that the first term is also 0. If $j=0$, then $s^jd^0 = id$. We thus have

$$s^j\beta^n_{(k)}(c\otimes\Sigma^{-1}\underline{0\cdots n}) = (-1)^{|c|}H^0_{n-1}\beta^{n-1}_{(k)}(c\otimes\Sigma^{-1}\underline{0\cdots(n-1)}).$$

By definition of $\beta^{n-1}_{(k)}$, the term $\beta^{n-1}_{(k)}(c\otimes\Sigma^{-1}\underline{0\cdots(n-1)})$ is in the image of H^0_{n-1}. Since $H^0_{n-1}H^0_{n-1}=0$, we obtain $s^j\beta^n_{(k)}(c\otimes\Sigma^{-1}\underline{0\cdots n}) = 0$. We thus have proved that $s^j\beta^n_{(k)}(c\otimes\Sigma^{-1}\underline{0\cdots n}) = \beta^n_{(k)}s^j(c\otimes\Sigma^{-1}\underline{0\cdots n}) = 0$. Suppose now that $\underline{x}\neq\underline{0\cdots n}$. Then there exists $0\leq i\leq n$ such that $\underline{x} = \underline{0\ldots(i-1)a_i\cdots a_r}$ with $i<a_i<\cdots<a_r\leq n$. We thus have

$$s^j\beta^n_{(k)}(c\otimes\Sigma^{-1}\underline{x}) = s^jd^i\beta^{n-1}_{(k)}(c\otimes\Sigma^{-1}s^{i-1}\underline{x}).$$

If $i<j$, then $s^jd^i = d^is^{j-1}$ so that

$$s^j\beta^n_{(k)}(c\otimes\Sigma^{-1}\underline{x}) = d^is^{j-1}\beta^{n-1}_{(k)}(c\otimes\Sigma^{-1}s^{i-1}\underline{x}).$$

By induction hypothesis on $n-1$, we obtain

$$\begin{aligned} s^j\beta^n_{(k)}(c\otimes\Sigma^{-1}\underline{x}) &= d^i\beta^{n-2}_{(k)}(c\otimes\Sigma^{-1}s^{j-1}s^{i-1}\underline{x}) \\ &= d^i\beta^{n-2}_{(k)}(c\otimes\Sigma^{-1}s^{i-1}s^j\underline{x}) \\ &= \beta^n_{(k)}s^j(c\otimes\Sigma^{-1}\underline{x}). \end{aligned}$$

If $i=j,j+1$, then

$$s^j\beta^n_{(k)}(c\otimes\Sigma^{-1}\underline{x}) = \beta^{n-1}_{(k)}(c\otimes\Sigma^{-1}s^{i-1}\underline{x}).$$

Since we have $s^{i-1}\underline{x} = s^i\underline{x}$, in any case, this gives

$$s^j\beta^n_{(k)}(c\otimes\Sigma^{-1}\underline{x}) = \beta^{n-1}_{(k)}s^j(c\otimes\Sigma^{-1}\underline{x}).$$

If $i > j+1$, then, by induction hypothesis on $n-1$,

$$\begin{array}{rcl} s^j\beta^n_{(k)}(c\otimes\Sigma^{-1}\underline{x}) &=& d^{i-1}\beta^{n-2}_{(k)}(c\otimes\Sigma^{-1}s^js^{i-1}\underline{x}) \\ &=& d^{i-1}\beta^{n-2}_{(k)}(c\otimes\Sigma^{-1}s^{i-2}s^j\underline{x}) \\ &=& 0 \\ &=& \beta^n_{(k)}s^j(c\otimes\Sigma^{-1}\underline{x}), \end{array}$$

since $s^j\underline{x}=0$. At the end, we have proved that $s^j\beta^n_{(k)}=\beta^{n-1}_{(k)}s^j$ and thus the lemma. $\square$

Remark 2.6.12. *In particular, this lemma implies that*

$$\beta^n_{(k)}(c\otimes\Sigma^{-1}\underline{0\cdots n}) = -\sum_{\substack{p+q=k\\ p\neq 0}} H^0_n\partial^n_{(p)}\beta^n_{(q)}(c\otimes\Sigma^{-1}\underline{0\cdots n})$$

for every $n,k\geq 0$ and $c\in\overline{\mathcal{C}_{ns}}$. Indeed, we have

$$\begin{array}{rcl} \partial^n_{(k)}\beta^n_{(0)}(c\otimes\Sigma^{-1}\underline{0\cdots n}) &=& -(-1)^{|c|}\sum_{i=0}^n(-1)^i\beta^n_{(k)}(c\otimes\Sigma^{-1}\underline{0\cdots\widehat{i}\cdots n}) \\ && +\beta^n_{(k)}(d_{\mathbf{Sur}_{\mathbb{K}}}(c)\otimes\Sigma^{-1}\underline{0\cdots n}) \\ &=& -(-1)^{|c|}\sum_{i=0}^n(-1)^i d^i\beta^{n-1}_{(k)}(c\otimes\Sigma^{-1}\underline{0\cdots(n-1)}) \\ && +\beta^n_{(k)}(d_{\mathbf{Sur}_{\mathbb{K}}}(c)\otimes\Sigma^{-1}\underline{0\cdots n}). \end{array}$$

By an immediate computation, if $i\neq 0$, then $d^iH^0_{n-1}=H^0_nd^i$. Since $\beta^{n-1}_{(k)}(c\otimes\Sigma^{-1}\underline{0\cdots(n-1)})$ is in the image of H^0_{n-1} by construction, and that $H^0_nH^0_n=0$, we obtain

$$\begin{array}{l} H^0_n\partial^n_{(k)}\beta^n_{(0)}(c\otimes\Sigma^{-1}\underline{0\cdots n}) \\ \quad = -(-1)^{|c|}H^0_nd^0\beta^{n-1}_{(k)}(c\otimes\Sigma^{-1}\underline{0\cdots(n-1)}) + H^0_n\beta^n_{(k)}(d_{\mathbf{Sur}_{\mathbb{K}}}(c)\otimes\Sigma^{-1}\underline{0\cdots n}) \end{array}$$

which proves the above formula.

Theorem 2.6.13. *Let $n\geq 0$. The morphism $\partial^n : \mathcal{F}(\overline{\mathcal{C}}\underset{\mathrm{H}}{\otimes}\overline{\mathbf{Sur}_{\mathbb{K}}}\otimes\Sigma^{-1}N_*(\Delta^n))\longrightarrow\mathcal{F}(\overline{\mathcal{C}}\underset{\mathrm{H}}{\otimes}\overline{\mathbf{Sur}_{\mathbb{K}}}\otimes\Sigma^{-1}N_*(\Delta^n))$ is such that $d_{\mathcal{C}}+\partial^n$ is a derivation of operads. Moreover, the sequence $\partial^\bullet : \mathcal{F}(\overline{\mathcal{C}}\underset{\mathrm{H}}{\otimes}\overline{\mathbf{Sur}_{\mathbb{K}}}\otimes\Sigma^{-1}N_*(\Delta^\bullet))\longrightarrow\mathcal{F}(\overline{\mathcal{C}}\underset{\mathrm{H}}{\otimes}\overline{\mathbf{Sur}_{\mathbb{K}}}\otimes\Sigma^{-1}N_*(\Delta^\bullet))$ is a morphism of cosimplicial sets.*

Proof. By Lemma 2.6.11, the morphisms $\partial^\bullet$ preserve the cosimplicial structure of $\mathcal{F}(\overline{\mathcal{C}}\underset{\mathrm{H}}{\otimes}\overline{\mathbf{Sur}_{\mathbb{K}}}\otimes\Sigma^{-1}N_*(\Delta^\bullet))$. We need to prove that $d_{\mathcal{C}}+\partial^n$ is a derivation of operads. This is equivalent to prove that

$$d_{\mathcal{C}}(\beta^n)+\partial^n\beta^n=0$$

for every $n\geq 0$. By an immediate induction, we have $d_{\mathcal{C}}(\beta^n)=0$. It remains to prove that $\partial^n\beta^n=0$. This is equivalent to prove that

$$\sum_{p+q=k}\partial^n_{(p)}\beta^n_{(q)}=0$$

for every $k \geq 0$. We prove it on $\overline{\mathcal{C}_{ns}} \otimes \Sigma^{-1} N_*(\Delta^n)$ by induction on $n, k \geq 0$, since all the maps are symmetric by construction. It is true for $n = 0$, and for $n \geq 1$ and $k = 0$. We now suppose that $n \geq 1$ and $k \geq 1$. Let $c \in \overline{\mathcal{C}_{ns}}$ and $\underline{x} \in N_*(\Delta^n)$ be a basis element. If $\underline{x} \neq \underline{0 \cdots n}$, then there exists $0 \leq i \leq n-1$ and $\underline{y} \in N_*(\Delta^{n-1})$ such that $d^i \underline{y} = \underline{x}$. Since the morphisms $d_{\mathcal{C}} + \partial^\bullet$ are compatible with the cosimplicial structure of $\mathcal{F}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}_{\mathbb{K}}} \otimes \Sigma^{-1} N_*(\Delta^n))$ by Lemma 2.6.11, we have

$$\sum_{p+q=k} \partial^n_{(p)} \beta^n_{(q)}(c \otimes \Sigma^{-1} \underline{x}) = \sum_{p+q=k} d^i \partial^{n-1}_{(p)} \beta^{n-1}_{(q)}(c \otimes \Sigma^{-1} \underline{y})$$

which is 0 by induction hypothesis on $n-1$. Suppose now that $\underline{x} = \underline{0 \cdots n}$. By using that

$$\partial^n_{(0)} H^0_n + H^0_n \partial^n_{(0)} = id - \Phi^0_n,$$

we have

$$\begin{aligned}
\partial^n_{(0)} \beta^n_{(k)}(c \otimes \Sigma^{-1} \underline{0 \cdots n}) = & \sum_{\substack{p+q=k \\ p \neq 0}} H^0_n \partial^n_{(0)} \partial^n_{(p)} \beta^n_{(q)}(c \otimes \Sigma^{-1} \underline{0 \cdots n}) \\
& - \sum_{\substack{p+q=k \\ p \neq 0}} \partial^n_{(p)} \beta^n_{(q)}(c \otimes \Sigma^{-1} \underline{0 \cdots n}) + \sum_{\substack{p+q=k \\ p \neq 0}} \Phi^0_n \partial^n_{(p)} \beta^n_{(q)}(c \otimes \Sigma^{-1} \underline{0 \cdots n}).
\end{aligned}$$

By Lemma 2.6.11, the morphism Φ^0_n commutes with the $\partial^n_{(p)}$'s. We thus have that the last sum is 0, since $\phi^0_n(\underline{0 \cdots n}) = 0$ because $n \geq 1$. Now, we claim that

$$\sum_{\substack{p+q=k \\ p \neq 0}} \partial^n_{(0)} \partial^n_{(p)} \beta^n_{(q)}(c \otimes \Sigma^{-1} \underline{0 \cdots n}) = 0.$$

We write

$$\sum_{\substack{p+q=k \\ p \neq 0}} \partial^n_{(0)} \partial^n_{(p)} \beta^n_{(q)}(c \otimes \Sigma^{-1} \underline{0 \cdots n}) = \partial^n_{(0)} \partial^n_{(k)} \beta^n_{(0)}(c \otimes \Sigma^{-1} \underline{0 \cdots n}) + \sum_{\substack{p+q=k \\ p,q \neq 0}} \partial^n_{(0)} \partial^n_{(p)} \beta^n_{(q)}(c \otimes \Sigma^{-1} \underline{0 \cdots n}).$$

We first deal with the sum at the right hand-side. Since $p < k$, we can use our induction hypothesis on p to obtain

$$\sum_{\substack{p+q=k \\ p,q \neq 0}} \partial^n_{(0)} \partial^n_{(p)} \beta^n_{(q)}(c \otimes \Sigma^{-1} \underline{0 \cdots n}) = - \sum_{\substack{p+q=k \\ p,q \neq 0}} \sum_{\substack{s+t=p \\ s \neq 0}} \partial^n_{(s)} \partial^n_{(t)} \beta^n_{(q)}(c \otimes \Sigma^{-1} \underline{0 \cdots n}).$$

By a variable substitution, this gives

$$\sum_{\substack{p+q=k \\ p,q \neq 0}} \partial^n_{(0)} \partial^n_{(p)} \beta^n_{(q)}(c \otimes \Sigma^{-1} \underline{0 \cdots n}) = - \sum_{\substack{s+t=k \\ s,t \neq 0}} \partial^n_{(s)} \left(\sum_{\substack{p+q=t \\ q \neq 0}} \partial^n_{(p)} \beta^n_{(q)}(c \otimes \Sigma^{-1} \underline{0 \cdots n}) \right).$$

Now, since applying $\beta^n_{(0)}$ on $c \otimes \Sigma^{-1} \underline{0 \cdots n}$ allows us to apply our induction hypothesis, we have

$$\partial^n_{(0)}\partial^n_{(k)}\beta^n_{(0)}(c\otimes\Sigma^{-1}\underline{0\cdots n}) = -\sum_{\substack{s+t=k\\ s,t\neq 0}}\partial^n_{(s)}\partial^n_{(t)}\beta^n_{(0)}(c\otimes\Sigma^{-1}\underline{0\cdots n}).$$

At the end, we obtain that

$$\sum_{\substack{p+q=k\\ p,q\neq 0}}\partial^n_{(0)}\partial^n_{(p)}\beta^n_{(q)}(c\otimes\Sigma^{-1}\underline{0\cdots n}) = -\sum_{\substack{s+t=k\\ s,t\neq 0}}\partial^n_{(s)}\left(\sum_{p+q=t}\partial^n_{(p)}\beta^n_{(q)}(c\otimes\Sigma^{-1}\underline{0\cdots n})\right),$$

and this last sum is 0 by induction hypothesis on $t<k$. We thus have proved that

$$\partial^n_{(0)}\beta^n_{(k)}(c\otimes\Sigma^{-1}\underline{0\cdots n}) = -\sum_{\substack{p+q=k\\ p\neq 0}}\partial^n_{(p)}\beta^n_{(q)}(c\otimes\Sigma^{-1}\underline{0\cdots n}),$$

which is equivalent to

$$\sum_{p+q=k}\partial^n_{(p)}\beta^n_{(q)}(c\otimes\Sigma^{-1}\underline{0\cdots n}) = 0.$$

The theorem is proved. □

Theorem 2.6.14. *Let $\mathcal{C}$ be a symmetric cooperad. Then*

$$B^c(\mathcal{C}\otimes\mathbf{Sur}_{\mathbb{K}})\otimes\Delta^\bullet := (\mathcal{F}(\overline{\mathcal{C}}\underset{\mathrm{H}}{\otimes}\overline{\mathbf{Sur}}_{\mathbb{K}}\otimes\Sigma^{-1}N_*(\Delta^\bullet)),\partial^\bullet)$$

where $\partial^\bullet$ is the twisting derivation constructed in Theorem 2.6.13 is a cosimplicial frame associated to $B^c(\mathcal{C})$.

Proof. The proof uses the same arguments as Theorem 2.5.11, with the differentials constructed in Theorem 2.6.13. □

2.6.3 Computation of $\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}),\mathcal{P})$

We now describe a mapping space $\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}\underset{\mathrm{H}}{\otimes}\mathbf{Sur}_{\mathbb{K}}),\mathcal{P})$ for some coaugmented connected cooperad $\mathcal{C}$ and for some augmented connected operad $\mathcal{P}$. We recall the following definition.

Definition 2.6.15. *Let $M\in\Sigma\mathrm{Seq}^0_{\mathbb{K}}$ be an augmented sequence $M\simeq I\oplus\overline{M}$ with differential d. The sequence M is an* operad up to homotopy *if there exists a derivation of cooperads of the form $d+\partial$ on $\mathcal{F}^c(\Sigma\overline{M})$ with $\partial_{|\Sigma\overline{M}}=0$.*

In this situation, we say that ∂ is a *twisting morphism*, and that $d+\partial$ is a *twisted derivation.* Recall that giving such a differential is equivalent to giving a morphism $\beta:\mathcal{F}^c(\Sigma\overline{M})\longrightarrow\Sigma\overline{M}$ such that $\beta_{|\Sigma\overline{M}}=0$ and, if we denote by ∂ the morphism obtained from β by the Leibniz rule on $\mathcal{F}^c(\Sigma\overline{M})$, then

$$d(\beta)+\beta\partial=0.$$

Proposition 2.6.16. *Let $M \in \Sigma\mathrm{Seq}^0_{\mathbb{K}}$ be an operad up to homotopy. Then $\mathcal{L}(M) := \bigoplus_{n\geq 2} M(n)^{\Sigma_n}$ is endowed with the structure of a $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebra.*

Proof. Let ∂ be the twisting part of the differential on $\mathcal{F}^c(\Sigma\overline{M})$. We denote by β its composite with the projection on $\Sigma\overline{M}$. Let $x, y_1, \ldots, y_n \in \mathcal{L}(M)$ be elements with homogeneous degrees and arities, and $r_1, \ldots, r_n \geq 0$. We let E to be the symmetric sequence spanned by abstract invariant variables $Y_1, \ldots, Y_n, dY_1, \ldots, dY_n$ of the same arities and degrees as $y_1, \ldots, y_n, d(y_1), \ldots, d(y_n)$. Let $\psi : E \longrightarrow \Sigma\overline{M}$ be the morphism which sends the Y_i's to the y_i's. This gives a morphism of coalgebras $\psi : \Gamma(\mathcal{L}(E)) \longrightarrow \Gamma(\Sigma\mathcal{L}(M))$. Let $r_1, \ldots, r_n \geq 0$ be such that $r := r_1 + \cdots + r_n \neq 0$ (with, for every $1 \leq i \leq n$, the assumption that $r_i = 1$ if Y_i has an odd degree). We set $x\{\!\{\,\}\!\} = d(x)$ and

$$x\{\!\{y_1, \ldots, y_n\}\!\}_{r_1,\ldots,r_n} = \sum_{\underline{T}\in\mathcal{TM}(r+1)} \beta\left(\underline{T}^\vee \otimes x \otimes \psi\mathcal{O}(Y_1^{r_1} \cdots Y_n^{r_n})\right),$$

where we consider the orbit map $\mathcal{O}$ defined in Proposition 2.2.20 and where, in this sum, we identify every tensor $\underline{T}^\vee \otimes z$ such that $z \notin \bigotimes_{i=1}^{r+1} \Sigma\overline{M}(\mathrm{val}_{\underline{T}}(i))$ with 0. In particular, the above sum is finite.

We first note that these operations preserve $\mathcal{L}(M)$. Indeed, the symmetry relations in the cooperad $\mathcal{F}^c(\Sigma\overline{M})$ will only make involve either actions of symmetric groups elements on x and the Y_i's, which are invariant, or actions on the tensors given by $\mathcal{O}(Y_1^{\otimes r_1} \cdots Y_n^{\otimes r_n})$, which is invariant under the action of Σ_r. It remains to prove formulas of Theorem 2.2.22. It is an immediate check that the operations $-\{\!\{-, \ldots, -\}\!\}_{r_1,\ldots,r_n}$ satisfy the relations $(i) - (v)$. We now check relation (vi). First, we have

$$\sum_{\underline{T}\in\mathcal{TM}(r+1)} d(\beta(\underline{T}^\vee \otimes x \otimes \psi\mathcal{O}(Y_1^{r_1} \cdots Y_n^{r_n}))) = x\{\!\{y_1, \ldots, y_n\}\!\}_{r_1,\ldots,r_n}\{\!\{\,\}\!\};$$

$$\sum_{\underline{T}\in\mathcal{TM}(r+1)} \beta(\underline{T}^\vee \otimes d(x) \otimes \psi\mathcal{O}(Y_1^{r_1} \cdots Y_n^{r_n})) = x\{\!\{\,\}\!\}\{\!\{y_1, \ldots, y_n\}\!\}_{r_1,\ldots,r_n};$$

and

$$\sum_{\underline{T}\in\mathcal{TM}(r+1)} \beta(\underline{T}^\vee \otimes x \otimes \psi\mathcal{O}(dY_k \cdot Y_1^{r_1} \cdots Y_k^{r_k - 1} \cdots Y_n^{r_n}))$$
$$= x\{\!\{y_k\{\!\{\,\}\!\}, y_1, \ldots, y_n\}\!\}_{1,r_1,\ldots,r_k-1,\ldots,r_n}.$$

for every $1 \leq k \leq n$. This gives

$$\sum_{\underline{T}\in\mathcal{TM}(r+1)} d(\beta)(\underline{T}^\vee \otimes x \otimes \psi\mathcal{O}(Y_1^{r_1} \cdots Y_n^{r_n}))$$
$$= x\{\!\{y_1, \ldots, y_n\}\!\}_{r_1,\ldots,r_n}\{\!\{\,\}\!\} + x\{\!\{\,\}\!\}\{\!\{y_1, \ldots, y_n\}\!\}_{r_1,\ldots,r_n}$$
$$+ \sum_{k=1}^n \pm x\{\!\{y_k\{\!\{\,\}\!\}, y_1, \ldots, y_n\}\!\}_{1,r_1,\ldots,r_k-1,\ldots,r_n}.$$

Let $\Delta : \mathcal{F}^c(\Sigma\overline{M}) \longrightarrow \mathcal{F}^c(\mathcal{F}^c(\Sigma\overline{M}))$ be the cooperad morphism induced by the cooperad structure of $\mathcal{F}^c(\Sigma\overline{M})$. Let $\pi_{\Sigma\overline{M}} : \mathcal{F}^c(\Sigma\overline{M}) \longrightarrow \Sigma\overline{M}$ be the projection on $\Sigma\overline{M}$. We keep the notation ∂ for the morphism defined on $\mathcal{F}^c(\mathcal{F}^c(\Sigma\overline{M}))$ obtained from ∂ by the Leibniz rule. Since ∂ is compatible with the cooperad structure on $\mathcal{F}^c(\Sigma\overline{M})$, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{F}^c(\Sigma\overline{M}) \arrow[r, "\Delta"] \arrow[d, "\partial"'] & \mathcal{F}^c(\mathcal{F}^c(\Sigma\overline{M})) \arrow[d, "\partial"] \\
\mathcal{F}^c(\Sigma\overline{M}) \arrow[r, "\Delta"'] \arrow[dr, no head, "="'] & \mathcal{F}^c(\mathcal{F}^c(\Sigma\overline{M})) \arrow[d, "\mathcal{F}^c(\pi_{\Sigma\overline{M}})"] \\
& \mathcal{F}^c(\Sigma\overline{M})
\end{tikzcd} .$$

Let $\mathcal{F}^c_{(\geq 2)}(\Sigma\overline{M})$ be the sub symmetric sequence of $\mathcal{F}^c(\Sigma\overline{M})$ given by trees with at least 2 vertices, so that $\mathcal{F}^c(\Sigma\overline{M}) \simeq \Sigma\overline{M} \oplus \mathcal{F}^c_{(\geq 2)}(\Sigma\overline{M})$ as a symmetric sequence. We denote by $\mathcal{F}^c(\Sigma\overline{M}; \mathcal{F}^c_{(\geq 2)}(\Sigma\overline{M}))$ the sub symmetric sequence of $\mathcal{F}^c(\mathcal{F}^c(\Sigma\overline{M}))$ given by trees with only one vertex in $\mathcal{F}^c_{(\geq 2)}(\Sigma\overline{M})$, and the other in $\Sigma\overline{M}$. Then we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{F}^c(\mathcal{F}^c(\Sigma\overline{M})) \arrow[r, two heads] \arrow[d, "\partial"'] & \mathcal{F}^c(\Sigma\overline{M}; \mathcal{F}^c_{(\geq 2)}(\Sigma\overline{M})) \arrow[d, "\mathcal{F}^c(\Sigma\overline{M};\partial)"] \\
\mathcal{F}^c(\mathcal{F}^c(\Sigma\overline{M})) \arrow[d, two heads, "\mathcal{F}^c(\pi_{\Sigma\overline{M}})"'] & \mathcal{F}^c(\Sigma\overline{M}; \mathcal{F}^c(\Sigma\overline{M})) \arrow[d, "\mathcal{F}^c(\Sigma\overline{M};\pi_{\Sigma\overline{M}})"] \\
\mathcal{F}^c(\Sigma\overline{M}) & \mathcal{F}^c(\Sigma\overline{M}; \Sigma\overline{M}) \arrow[l, two heads, "\mathcal{F}^c(id_{\Sigma\overline{M}} \oplus id_{\Sigma\overline{M}})"]
\end{tikzcd} .$$

The above commutative diagrams prove that $\partial : \mathcal{F}^c_{(\geq 2)}(\Sigma\overline{M}) \longrightarrow \mathcal{F}^c(\Sigma\overline{M})$ is given by the composite:

$$\partial : \mathcal{F}^c_{(\geq 2)}(\Sigma\overline{M}) \xrightarrow{\Delta_{(1)}} \mathcal{F}^c(\Sigma\overline{M}; \mathcal{F}^c_{(\geq 2)}(\Sigma\overline{M})) \xrightarrow{\mathcal{F}^c(\Sigma\overline{M};\beta)} \mathcal{F}^c(\Sigma\overline{M}; \Sigma\overline{M}) \xrightarrow{\mathcal{F}^c(id_{\Sigma\overline{M}} \oplus id_{\Sigma\overline{M}})} \mathcal{F}^c(\Sigma\overline{M})$$

where $\Delta_{(1)}$ is the composite of $\Delta : \mathcal{F}^c_{(\geq 2)}(\Sigma\overline{M}) \longrightarrow \mathcal{F}^c(\mathcal{F}^c(\Sigma\overline{M}))$ with the projection on trees with only one vertex in $\mathcal{F}^c_{(\geq 2)}(\Sigma\overline{M})$. By definition, for every $\underline{T} \in \mathcal{TM}(r+1)$, we have

$$\begin{aligned}
&\Delta_{(1)}(\underline{T}^\vee \otimes x \otimes \psi\mathcal{O}(Y_1^{r_1}\cdots Y_n^{r_n}))\\
&= \sum_{\substack{p+q=r\\ p\neq 0}} \sum_{\substack{\underline{U}\in\mathcal{TM}(q+1)\\ \underline{V}\in\mathcal{TM}(p+1)\\ \underline{U}\bullet_1\underline{V}=\underline{T}}} \sum_{\substack{p_i+q_i=r_i\\ p_1+\cdots+p_n=p\\ q_1+\cdots+q_n=q}} \pm\underline{U}^\vee \otimes (\underline{V}^\vee \otimes x \otimes \psi\mathcal{O}(Y_1^{p_1}\cdots Y_n^{p_n})) \otimes \psi\mathcal{O}(Y_1^{q_1}\cdots Y_n^{q_n})\\
&\qquad + \sum_{\substack{p+q=r-1\\ p\neq 0}} \sum_{k=0}^{q} \sum_{\substack{\underline{U}\in\mathcal{TM}(q+1)\\ \underline{V}\in\mathcal{TM}(p+1)\\ \underline{U}\bullet_k\underline{V}=\underline{T}}} \sum_{\substack{s_i+p'_i+t_i=r_i\\ p'_1+\cdots+p'_n=p+1\\ s_1+\cdots+s_n=k\\ t_1+\cdots+t_n=q-k}} \pm\underline{U}^\vee \otimes x \otimes \psi\mathcal{O}(Y_1^{s_1}\cdots Y_n^{s_n})\\
&\qquad\qquad \otimes \left(\underline{V}^\vee \otimes \psi\mathcal{O}(Y_1^{p'_1}\cdots Y_n^{p'_n})\right) \otimes \psi\mathcal{O}(Y_1^{t_1}\cdots Y_n^{t_n}).
\end{aligned}$$

By some variable substitutions, summing over $\underline{T}\in\mathcal{TM}(r+1)$ gives

$$\begin{aligned}
&\sum_{\substack{p+q=r\\ p\neq 0}} \sum_{\substack{p_i+q_i=r_i\\ p_1+\cdots+p_n=p\\ q_1+\cdots+q_n=q}} \sum_{\underline{U}\in\mathcal{TM}(q+1)} \pm\underline{U}^\vee\otimes\left(\sum_{\underline{V}\in\mathcal{TM}(p+1)} \underline{V}^\vee \otimes x \otimes \psi\mathcal{O}(Y_1^{p_1}\cdots Y_n^{p_n})\right)\otimes\psi\mathcal{O}(Y_1^{q_1}\cdots Y_n^{q_n})\\
&\qquad + \sum_{\substack{p+q=r-1\\ p\neq 0}} \sum_{\substack{p'_i+q_i=r_i\\ p'_1+\cdots+p'_n=p+1\\ q_1+\cdots+q_n=q}} \sum_{\underline{U}\in\mathcal{TM}(q+1)} \pm\underline{U}^\vee \otimes x\\
&\qquad \otimes \mathrm{Sh}\left(\left(\sum_{\underline{V}\in\mathcal{TM}(p+1)} \underline{V}^\vee \otimes \psi\mathcal{O}(Y_1^{p'_1}\cdots Y_n^{p'_n})\right); \psi\mathcal{O}(Y_1^{q_1}\cdots Y_n^{q_n})\right),
\end{aligned}$$

where Sh is defined analogously as in Definition 2.2.14. We now use that

$$\psi\mathcal{O}(Y_1^{p'_1}\cdots Y_n^{p'_n}) = \sum_{k=1}^{n} \pm y_k \otimes \psi\mathcal{O}(Y_1^{p'_1}\cdots Y_k^{p'_k-1}\cdots Y_n^{p'_n})$$

for every $p'_1,\ldots,p'_n \geq 0$ (if $p'_k=0$ for some k, we just remove the corresponding term). For a chosed $1\leq k\leq n$, we set $p_i=p'_i$ for every $i\neq k$ and $p_k=p'_k-1$. Then, applying $\mathcal{F}^c(\Sigma\overline{M};\beta)$, gives

$$\begin{aligned}
&\sum_{\substack{p+q=r\\ p\neq 0}} \sum_{\substack{p_i+q_i=r_i\\ p_1+\cdots+p_n=p\\ q_1+\cdots+q_n=q}} \sum_{\underline{U}\in\mathcal{TM}(q+1)} \pm\underline{U}^\vee \otimes (x\{\!\!\{y_1,\ldots,y_n\}\!\!\}_{p_1,\ldots,p_n}) \otimes \psi\mathcal{O}(Y_1^{q_1}\cdots Y_n^{q_n})\\
&+ \sum_{\substack{p+q=r-1\\ p\neq 0}} \sum_{k=1}^{n} \sum_{\substack{p_i+q_i=r_i, i\neq k\\ p_k+q_k=r_k-1\\ p_1+\cdots+p_n=p\\ q_1+\cdots+q_n=q}} \sum_{\underline{U}\in\mathcal{TM}(q+1)} \pm\underline{U}^\vee\otimes x\otimes\mathrm{Sh}\left(y_k\{\!\!\{y_1,\ldots,y_n\}\!\!\}_{p_1,\ldots,p_n}; \psi\mathcal{O}(Y_1^{q_1}\cdots Y_n^{q_n})\right).
\end{aligned}$$

Applying β again gives

$$\sum_{\substack{p_i+q_i=r_i\\ p_1+\cdots+p_n\neq 0\\ q_1+\cdots+q_n\neq 0}} \pm x\{\!\{y_1,\ldots,y_n\}\!\}_{p_1,\ldots,p_n}\{\!\{y_1,\ldots,y_n\}\!\}_{q_1,\ldots,q_n}$$
$$+\sum_{k=1}^{n}\sum_{\substack{p_i+q_i=r_i, i\neq k\\ p_k+q_k=r_k-1\\ p_1+\cdots+p_n\neq 0}} \pm x\{\!\{y_k\{\!\{y_1,\ldots,y_n\}\!\}_{p_1,\ldots,p_n}, y_1,\ldots,y_n\}\!\}_{1,q_1,\ldots,q_n}.$$

Finally, the equation $d(\beta)+\beta\partial = 0$ applied on $\sum_{\underline{T}\in\mathcal{TM}(r+1)}\underline{T}\otimes x\otimes\psi\mathcal{O}(Y_1^{r_1}\cdots Y_n^{r_n})$ gives

$$\sum_{p_i+q_i=r_i} \pm x\{\!\{y_1,\ldots,y_n\}\!\}_{p_1,\ldots,p_n}\{\!\{y_1,\ldots,y_n\}\!\}_{q_1,\ldots,q_n}$$
$$+\sum_{k=1}^{n}\sum_{\substack{p_i+q_i=r_i, i\neq k\\ p_k+q_k=r_k-1}} \pm x\{\!\{y_k\{\!\{y_1,\ldots,y_n\}\!\}_{p_1,\ldots,p_n}, y_1,\ldots,y_n\}\!\}_{1,q_1,\ldots,q_n} = 0$$

as desired. □

Corollary 2.6.17. *Let $M\in\Sigma\mathrm{Seq}^0_{\mathbb{K}}$ be an operad up to homotopy. Then the completion of $\mathcal{L}(\Sigma M)$, which is $\prod_{n\geq 1}\Sigma M(n)^{\Sigma_n}$, is endowed with the structure of a complete $\Gamma\Lambda\mathcal{PL}_\infty$-algebra.*

Proof. It is the same proof as for [Ver23, Corollary 2.18]. □

We now apply this proposition to $M=\mathrm{Hom}(\mathcal{C}\underset{\mathrm{H}}{\otimes}\mathbf{Sur}_{\mathbb{K}}\otimes N_*(\Delta^n),\mathcal{P})$ for every $n\geq 0$, which will give Theorem H.

Theorem 2.6.18. *Let $\mathcal{C}$ be a symmetric cooperad and $\mathcal{P}$ be a symmetric augmented operad such that $\mathcal{P}(0)=\mathcal{C}(0)=0$ and $\mathcal{P}(1)=\mathcal{C}(1)=\mathbb{K}$. Then, for every $n\geq 0$, the symmetric sequence $\mathrm{Hom}(\mathcal{C}\underset{\mathrm{H}}{\otimes}\mathbf{Sur}_{\mathbb{K}}\otimes N_*(\Delta^n),\mathcal{P})$ is an operad up to homotopy such that the underlying $\Gamma\widehat{\Lambda\mathcal{PL}_\infty}$-algebra structure on $\Sigma\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}\underset{\mathrm{H}}{\otimes}\overline{\mathbf{Sur}}_{\mathbb{K}}\otimes N_*(\Delta^n),\overline{\mathcal{P}})$ satisfies*

$$\mathrm{Map}^h_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}),\mathcal{P})\simeq\mathcal{MC}(\Sigma\mathrm{Hom}_{\Sigma\mathrm{Seq}_{\mathbb{K}}}(\overline{\mathcal{C}}\underset{\mathrm{H}}{\otimes}\overline{\mathbf{Sur}}_{\mathbb{K}}\otimes N_*(\Delta^\bullet),\overline{\mathcal{P}})),$$

where we have set $\mathrm{Map}^h_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}),\mathcal{P})=\mathrm{Map}_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}\otimes\mathbf{Sur}_{\mathbb{K}}),\mathcal{P})$

Proof. Let $n\geq 0$. We first note that we have an isomorphism

$$\Sigma\mathrm{Hom}(\overline{\mathcal{C}}\underset{\mathrm{H}}{\otimes}\overline{\mathbf{Sur}}_{\mathbb{K}}\otimes N_*(\Delta^n),\mathcal{P})\simeq\mathrm{Hom}(\overline{\mathcal{C}}\underset{\mathrm{H}}{\otimes}\overline{\mathbf{Sur}}_{\mathbb{K}}\otimes\Sigma^{-1}N_*(\Delta^n),\mathcal{P}).$$

We thus need to construct a morphism

$$\beta:\mathcal{F}^c_{(\geq 2)}(\mathrm{Hom}(\overline{\mathcal{C}}\underset{\mathrm{H}}{\otimes}\overline{\mathbf{Sur}}_{\mathbb{K}}\otimes\Sigma^{-1}N_*(\Delta^n),\overline{\mathcal{P}}))\longrightarrow\mathrm{Hom}(\overline{\mathcal{C}}\underset{\mathrm{H}}{\otimes}\overline{\mathbf{Sur}}_{\mathbb{K}}\otimes\Sigma^{-1}N_*(\Delta^n),\overline{\mathcal{P}})$$

such that, if we denote by d the differential induced by the internal differential of $\mathrm{Hom}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1}N_*(\Delta^n), \overline{\mathcal{P}})$, and if we denote by $\partial : \mathcal{F}^c_{(\geq 2)}(\mathrm{Hom}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1}N_*(\Delta^n), \overline{\mathcal{P}})) \longrightarrow \mathcal{F}^c(\mathrm{Hom}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1}N_*(\Delta^n), \overline{\mathcal{P}}))$ the morphism obtained from β by the Leibniz rule, then $d(\beta) + \beta\partial = 0$.

We set, for every $f_1, \ldots, f_m \in \mathrm{Hom}(\overline{\mathcal{C}} \underset{\mathrm{H}}{\otimes} \overline{\mathbf{Sur}}_{\mathbb{K}} \otimes \Sigma^{-1}N_*(\Delta^n), \overline{\mathcal{P}})$ and $\underline{T} \in \mathcal{TM}(m)$,

$$\beta(\underline{T}^\vee \otimes f_1 \otimes \cdots \otimes f_m) = \gamma_{(\underline{T})} \circ (f_1 \otimes \cdots \otimes f_m) \circ \beta^n_{(\underline{T})}$$

where $\beta^n_{(\underline{T})}$ is the composite of the morphism β^n defined in Construction 2.6.10 with the projection on the $\underline{T}$-component. We first note that $d(\beta) = 0$, since $d(\beta^n) = 0$. Now, note that the $\underline{T}$-component of $\beta\partial(\underline{T}^\vee \otimes f_1 \otimes \cdots \otimes f_m)$ is

$$\sum_{\underline{S} \subset \underline{T}} \gamma_{(\underline{T}/\underline{S})} \circ \left(f_1 \otimes \cdots \otimes f_{r(\underline{S})-1} \otimes \left(\gamma_{(\underline{S})} \circ \left(\bigotimes_{i \in V_{\underline{S}}} f_i \right) \circ \beta^n_{(\underline{S})} \right) \otimes \bigotimes_{i \in V_{\underline{T}/\underline{S}} \setminus \{1, \ldots, r(\underline{S})-1\}} f_i \right) \circ \beta^n_{(\underline{T}/\underline{S})}.$$

By Lemma 2.6.9, this is equal to

$$-\gamma_{(\underline{T})} \circ (f_1 \otimes \cdots \otimes f_m) \circ \left(\sum_{\underline{S} \subset \underline{T}} \beta^n_{(\underline{T}/\underline{S})} \circ_{\underline{S}} \beta^n_{(\underline{S})} \right).$$

This terms is 0, since the sum $\sum_{\underline{S} \subset \underline{T}} \beta^n_{(\underline{T}/\underline{S})} \circ_{\underline{S}} \beta^n_{(\underline{S})}$ is precisely the $\underline{T}$-component of $\partial^n \beta^n$, which is 0 by the proof of Theorem 2.6.13.

The computation of $\mathrm{Map}^h_{\Sigma\mathcal{O}p^0}(B^c(\mathcal{C}), \mathcal{P})$ comes from the construction of $B^c(\mathcal{C} \underset{\mathrm{H}}{\otimes} \mathbf{Sur}_{\mathbb{K}}) \otimes \Delta^\bullet$ given by Construction 2.6.10 and from the complete $\Gamma(\mathcal{P}re\mathcal{L}ie_\infty, -)$-algebra structure given by Corollary 2.6.17. $\square$

Appendix A

Symmetric sequences and operads

Outline of the current chapter

In this appendix, we recall the notion of an operad and its dual notion, the notion of a cooperad, and the basic concepts of the theory. The notion of an operad first appeared in the study of loop spaces. It has now become a powerful tool in order to study algebraic structures. Our main reference for operads theory is Fresse's book [Fre17a].

The notion of an operad can be defined in the context of a symmetric monoidal category. For simplicity, in this appendix, we work in a category of modules over a fixed ground ring $\mathbb{K}$, with $\otimes = \otimes_{\mathbb{K}}$. In this thesis, we mainly consider operads and algebras in dg $\mathbb{K}$-modules. In this context, we generalize the construction using the tensor products of dg $\mathbb{K}$-modules (see the conventions in Chapter 2 for a more comprehensive review of this background).

In §A.1, we give recollections and conventions on the symmetric groups.

In §A.2, we study the category of symmetric sequences $\Sigma\mathrm{Seq}_{\mathbb{K}}$ and its structures.

In §A.3, we define and study the notion of an operad.

A.1 Recollections on the symmetric groups

We begin this appendix by some definitions on permutations. In this appendix, and in all this thesis, we denote by Σ_n the symmetric groups on the elements $1,\dots,n$. We usually denote by id the relevant identity permutation, and we write any permutation $\sigma \in \Sigma_n$ as its sequence of values $(\sigma(1)\cdots\sigma(n))$. We also denote, for every $k,l \in \mathbb{N}$, by $[\![k;l]\!]$ the set of integers $n \in \mathbb{N}$ between k and l.

A.1.1 Direct sums and block permutations

Definition A.1.1. *Let $p,q \in \mathbb{N}$ and $\sigma \in \Sigma_p, \tau \in \Sigma_q$. We define $\sigma \oplus \tau \in \Sigma_{p+q}$ by*

$$\forall i \in [\![1;p]\!], (\sigma \oplus \tau)(i) = i,$$

$$\forall i \in [\![p+1;p+q]\!], (\sigma \oplus \tau)(i) = p + \tau(i).$$

In other words, $\sigma \oplus \tau$ acts as σ on the first p values in $[\![1;p+q]\!]$, and acts as τ for the following q values. For instance, if $\sigma = (132)$ and $\tau = (2431)$, then

$$\sigma \oplus \tau = (1325764).$$

We immediately see that the operation $\oplus$ is associative in $\bigsqcup_{n\geq 0} \Sigma_n$. This allows us to generalize the definition to the direct sum of r permutations $\sigma_1 \oplus \cdots \oplus \sigma_r$.

The direct sum behaves well with the group structure in Σ_n in the following sense.

Lemma A.1.2. *Let $\sigma_1, \tau_1 \in \Sigma_{n_1}, \sigma_2, \tau_2 \in \Sigma_{n_2}, \dots, \sigma_r, \tau_r \in \Sigma_{n_r}$. Then*

$$(\sigma_1 \oplus \cdots \oplus \sigma_r) \cdot (\tau_1 \oplus \cdots \oplus \tau_r) = \sigma_1\tau_1 \oplus \cdots \oplus \sigma_r\tau_r.$$

In particular, if we set $n = n_1 + \cdots + n_r$, then we have an inclusion of groups $\Sigma_{n_1} \times \cdots \times \Sigma_{n_r} \hookrightarrow \Sigma_n$ given by the direct sums.

We now define the notion of a *block permutation*.

Definition A.1.3. *Let $n_1,\dots,n_r \geq 1$, $n = n_1+\cdots+n_r$ and $\sigma \in \Sigma_r$. We set $\boldsymbol{n_i} = [\![n_1+\cdots+n_{i-1}+1; n_1+\cdots+n_i]\!]$. We define the* block permutation induced by σ and $(n_1,\dots,n_r)$ *by*

$$\sigma_*(n_1,\dots,n_r) = \boldsymbol{n}_{\sigma(1)} \cdots \boldsymbol{n}_{\sigma(r)}.$$

For instance, if we set $\sigma = (231) \in \Sigma_3$, then

$$\sigma_*(2,2,1) = (34512).$$

Lemma A.1.4. *Let $\sigma, \tau \in \Sigma_r$ and $n_1,\dots,n_r \geq 1$. Then*

$$\sigma_*(n_1,\dots,n_r) \cdot \tau_*(n_{\sigma(1)},\dots,n_{\sigma(r)}) = (\sigma\tau)_*(n_1,\dots,n_r).$$

We also have a commutation relation between the bloc permutations and the direct sum.

Lemma A.1.5. *Let $\sigma \in \Sigma_r$ and $\tau_1 \in \Sigma_{n_1}, \ldots, \tau_r \in \Sigma_{n_r}$. Then*

$$(\tau_1 \oplus \cdots \oplus \tau_r) \cdot \sigma_*(n_1, \ldots, n_r) = \sigma_*(n_1, \ldots, n_r) \cdot (\tau_{\sigma(1)} \oplus \cdots \oplus \tau_{\sigma(r)}).$$

This leads us to the following definition.

Definition A.1.6. *Let $\sigma \in \Sigma_r$ and $\tau_1 \in \Sigma_{n_1}, \ldots, \tau_r \in \Sigma_{n_r}$. We define the permutation $\sigma(\tau_1, \ldots, \tau_r) \in \Sigma_{n_1+\cdots+n_r}$ by*

$$\sigma(\tau_1, \ldots, \tau_r) = (\tau_1 \oplus \cdots \oplus \tau_r) \cdot \sigma_*(n_1, \ldots, n_r).$$

A.1.2 Shuffle and pointed shuffle permutations

In operad theory, we consider decompositions involving direct sums, blocks permutations and certain permutations called shuffle permutations.

Definition A.1.7. *Let $n = n_1 + \cdots + n_r$. A $(n_1, \ldots, n_r)$-*shuffle permutation *is a permutation in Σ_n which preserves the order on each block $[\![n_1 + \cdots + n_{i-1} + 1; n_1 + \cdots + n_i]\!]$. We denote by $Sh(n_1, \ldots, n_r)$ the set composed of such permutations.*
A shuffle permutation $\omega \in Sh(n_1, \ldots, n_r)$ is pointed *if it satisfies $\omega(1) < \omega(n_1 + 1) < \cdots < \omega(n_1 + \cdots + n_{r-1} + 1)$. We denote by $Sh_*(n_1, \ldots, n_r)$ the set composed of pointed $(n_1, \ldots, n_r)$-shuffle permutations.*

Proposition A.1.8. *Let $n \geq 0$ and $n_1, \ldots, n_r \geq 1$ such that $n_1 + \cdots + n_r = n$.*

- *Every $\sigma \in \Sigma_n$ admits a unique decomposition of the form*

$$\sigma = \omega \cdot (\tau_1 \oplus \cdots \oplus \tau_r)$$

  where $\tau_i \in \Sigma_{n_i}$ and $\omega \in Sh(n_1, \ldots, n_r)$.
- *Every $\sigma \in \Sigma_n$ admits a unique decomposition of the form*

$$\sigma = \omega \cdot \sigma(\tau_1, \ldots, \tau_r)$$

  where $\tau_i \in \Sigma_{n_i}, \sigma \in \Sigma_r$ and $\omega \in Sh_*(n_1, \ldots, n_r)$.

A.2 Symmetric sequences

A.2.1 Preliminary recollections

We first recall some definitions which will be useful in the context of symmetric sequences. We assume that G is any group. In this thesis, the group G will often be a subgroup of a symmetric group.

Definition A.2.1. *Let N be a left G-module.*

- *We denote by N^G the sub module of N given by elements $n \in N$ such that $g{\cdot}n = n$ for every $g \in G$.*

— *We denote by M_G the quotient of N by the relation $g \cdot n \equiv n$ for every $g \in G$ and $n \in N$.*

We have analogous definitions for right G-modules.

Recall that if M and N are left G-modules, then their tensor product $M \otimes N$ is a left G-module whose G-action is defined by

$$g \cdot (m \otimes n) = (g \cdot m) \otimes (g \cdot n)$$

for every $g \in G, m \in M$ and $n \in N$.

Definition A.2.2. *Let M and N be left G-modules.*

— *We define their* invariant tensor product *by $M \otimes^G N := (M \otimes N)^G$.*

— *We define their* coinvariant tensor product *by $M \otimes_G N := (M \otimes N)_G$.*

Remark A.2.3. *If M is a right G-module, then M is also a left G-module with as G-action $g \cdot m := m \cdot g^{-1}$ for every $g \in G$ and $m \in M$. We thus can apply the above definitions in this context.*

A.2.2 Definitions

Definition A.2.4. *A* symmetric sequence *(also called a Σ or $\mathbb{S}$-module) is a sequence of $\mathbb{K}$-modules $(M(n))_{n\in\mathbb{N}}$ such that Σ_n acts on $M(n)$ for every $n \geq 0$. A morphism of symmetric sequences $f : M \longrightarrow N$ is the data of a sequence of morphisms of modules $f_n : M(n) \longrightarrow N(n)$ which preserve the action of Σ_n.*

We denote the underlying category by $\Sigma\mathrm{Seq}_{\mathbb{K}}$.

This category admits several monoidal structures which are used in this thesis.

Definition A.2.5. *Let M and N be two symmetric sequences. We define their* tensor product *$M \otimes N$ as the symmetric sequence such that*

$$M \otimes N(n) = \bigoplus_{k+l=n} \mathbb{K}[\Sigma_n] \otimes_{\Sigma_k \times \Sigma_l} M(k) \otimes N(l),$$

where we make coincide the natural action of $\Sigma_k \times \Sigma_l$ on $M(k) \otimes N(l)$ with the action by right translation on $\Sigma_k \times \Sigma_l \hookrightarrow \Sigma_n$ on $\mathbb{K}[\Sigma_n]$. The Σ_n-action on $M \otimes N(n)$ is defined by the left translation of Σ_n on $\mathbb{K}[\Sigma_n]$.

Proposition A.2.6. *The map $(M, N) \longmapsto M \otimes N$ is a bifunctor which endows $\Sigma\mathrm{Seq}_{\mathbb{K}}$ with a structure of a symmetric monoidal category. The unit is given by $\mathbb{K}$ concentrated in arity 0. The symmetry operator is given by $\sigma \otimes m \otimes n \longmapsto \sigma \otimes n \otimes m$ for every $\sigma \in \Sigma_n$, $m \in M$ and $n \in N$.*

We define the composite of two symmetric sequences.

Definition A.2.7. *Let $M, N \in \Sigma\mathrm{Seq}_{\mathbb{K}}$.*

— *We define their* (coinvariant) composition *as*

$$M \circ N = \bigoplus_{r \geq 0} M(r) \otimes_{\Sigma_r} N^{\otimes r},$$

where we consider the action of Σ_r on $N^{\otimes r}$ by permutation of the factors.

— *We define their* invariant composition *as*

$$M \tilde{\circ} N = \bigoplus_{r \geq 0} M(r) \otimes^{\Sigma_r} N^{\otimes r}.$$

Remark A.2.8. *The composition of $M, N \in \Sigma\mathrm{Seq}_{\mathbb{K}}$ can be realized explicitly by using Proposition A.1.8:*

$$M \circ N(n) \simeq \bigoplus_{r \geq 0} \bigoplus_{i_1 + \cdots + i_r = n} \mathbb{K}[Sh_*(i_1, \ldots, i_r)] \otimes M(r) \otimes N(i_1) \otimes \cdots \otimes N(i_r).$$

Proposition A.2.9. *The bifunctors $\circ$ and $\tilde{\circ}$ endow $\Sigma\mathrm{Seq}_{\mathbb{K}}$ with the structure of a monoidal category with the unit I defined by:*

$$I(n) = \begin{cases} \mathbb{K} & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}.$$

The bifunctors $\circ$ and $\underset{\sim}{\circ}$ are related by the *trace map* $Tr_{M,N} : M \circ N \longrightarrow M \tilde{\circ} N$ defined by:

$$Tr_{M,N}(m \otimes n_1 \otimes \cdots \otimes n_r) = \sum_{\sigma \in \Sigma_r} (\sigma \cdot m) \otimes n_{\sigma^{-1}(1)} \otimes \cdots \otimes n_{\sigma^{-1}(r)}).$$

Proposition A.2.10 ([Fre00, §1.1.15])**.** *If $N(0) = 0$, then $Tr_{M,N}$ is an isomorphism.*

A.2.3 The tree representation

In practice, we often represent elements of a symmetric sequence by trees. This can be formalized by the following proposition.

Proposition A.2.11 ([Fre17a, Proposition 2.5.2])**.** *Let $\mathcal{B}ij$ be the category of finite sets with bijections as morphisms. Then giving a symmetric sequence $M \in \Sigma\mathrm{Seq}_{\mathbb{K}}$ is equivalent to giving a functor $M : \mathcal{B}ij \longrightarrow \mathrm{Mod}_{\mathbb{K}}$.*

Proof. Any symmetric sequence M gives rise to a functor $M : \mathcal{B}ij \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ defined by

$$M(I) = \mathcal{B}ij([\![1; n]\!], I) \otimes_{\Sigma_n} M(n).$$

In the converse direction, any functor $M : \mathcal{B}ij \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ gives a symmetric sequence defined by $M(n) := M([\![1; n]\!])$. □

Hence, every element $x \in M(n)$ can be seen as a tree

$$\begin{tikzcd}[column sep=small] i_1 \arrow[dr] & \cdots & i_n \arrow[dl] \\ & \boxed{x} \arrow[d] & \\ & 0 & \end{tikzcd} ,$$

where $I = \{i_1, \ldots, i_n\}$ is a finite set of n elements. This representation satisfies the relation

$$\begin{tikzcd}[column sep=small] i_1 \arrow[dr] & \cdots & i_n \arrow[dl] \\ & \boxed{\sigma \cdot x} \arrow[d] & \\ & 0 & \end{tikzcd} \equiv \begin{tikzcd}[column sep=small] i_{\sigma(1)} \arrow[dr] & \cdots & i_{\sigma(n)} \arrow[dl] \\ & \boxed{x} \arrow[d] & \\ & 0 & \end{tikzcd} .$$

We can also represent elements in a composite $M \circ N$ by a tree with two levels:

$$\begin{tikzcd}[column sep=tiny] i_1^1 \arrow[dr] & \cdots & i_{m_1}^1 \arrow[dl] & & i_1^n \arrow[dr] & \cdots & i_{m_n}^n \arrow[dl] \\ & \boxed{y_1} \arrow[drr] & & \cdots & & \boxed{y_n} \arrow[dll] & \\ & & & \boxed{x} \arrow[d] & & & \\ & & & 0 & & & \end{tikzcd} ,$$

where $x \in M(n)$ and $y_i \in M(m_i)$ for every $1 \le i \le n$. This representation is required to satisfy the following equivariance relations

$$\begin{tikzcd}[column sep=tiny] i_1^1 \arrow[dr] & \cdots & i_{m_1}^1 \arrow[dl] & & i_1^n \arrow[dr] & \cdots & i_{m_n}^n \arrow[dl] \\ & \boxed{\tau_1 \cdot y_1} \arrow[drr] & & \cdots & & \boxed{\tau_n \cdot y_n} \arrow[dll] & \\ & & & \boxed{x} \arrow[d] & & & \\ & & & 0 & & & \end{tikzcd} \equiv \begin{tikzcd}[column sep=tiny] i_{\tau_1(1)}^1 \arrow[dr] & \cdots & i_{\tau_1(m_1)}^1 \arrow[dl] & & i_{\tau_n(1)}^n \arrow[dr] & \cdots & i_{\tau_n(m_n)}^n \arrow[dl] \\ & \boxed{y_1} \arrow[drr] & & \cdots & & \boxed{y_n} \arrow[dll] & \\ & & & \boxed{x} \arrow[d] & & & \\ & & & 0 & & & \end{tikzcd} ;$$

$$\begin{tikzcd}[column sep=tiny] i_1^1 \arrow[dr] & \cdots & i_{m_1}^1 \arrow[dl] & & i_1^n \arrow[dr] & \cdots & i_{m_n}^n \arrow[dl] \\ & \boxed{y_1} \arrow[drr] & & \cdots & & \boxed{y_n} \arrow[dll] & \\ & & & \boxed{\sigma \cdot x} \arrow[d] & & & \\ & & & 0 & & & \end{tikzcd} \equiv \begin{tikzcd}[column sep=tiny] i_1^{\sigma(1)} \arrow[dr] & \cdots & i_{m_1}^{\sigma(1)} \arrow[dl] & & i_1^{\sigma(n)} \arrow[dr] & \cdots & i_{m_n}^{\sigma(n)} \arrow[dl] \\ & \boxed{y_{\sigma(1)}} \arrow[drr] & & \cdots & & \boxed{y_{\sigma(n)}} \arrow[dll] & \\ & & & \boxed{x} \arrow[d] & & & \\ & & & 0 & & & \end{tikzcd} ,$$

with $\sigma \in \Sigma_n$, $\tau_i \in \Sigma_{m_i}$ for all $1 \le i \le n$.

A.2.4 The functors $\mathcal{S}$ and Γ

Definition A.2.12. *Let M be a symmetric sequence and V be a $\mathbb{K}$-module. For every category $\mathcal{C}$, we denote by $\mathrm{End}(\mathcal{C})$ the category of endofunctors with natural transformations as morphisms. We define two functors $\mathcal{S}, \Gamma : \Sigma\mathrm{Seq}_{\mathbb{K}} \longrightarrow \mathrm{End}(\mathrm{Mod}_{\mathbb{K}})$ by*

$$\mathcal{S}(M, V) = \bigoplus_{n \ge 0} M(n) \otimes_{\Sigma_n} V^{\otimes n},$$

$$\Gamma(M,V) = \bigoplus_{n\geq 0} M(n) \otimes^{\Sigma_n} V^{\otimes n},$$

where we consider the action of Σ_n on $V^{\otimes n}$ given by the permutation of factors.

Proposition A.2.13. *Let $M, N \in \Sigma\mathrm{Seq}_{\mathbb{K}}$.*

— *We have natural isomorphisms*

$$\mathcal{S}(M \otimes N) \simeq \mathcal{S}(M) \otimes \mathcal{S}(N);$$

$$\Gamma(M \otimes N) \simeq \Gamma(M) \otimes \Gamma(N),$$

where we define the tensor product of functors pointwise by $(F \otimes G)(V) = F(V) \otimes G(V)$ for all functors $F, G \in \mathrm{End}(\mathrm{Mod}_{\mathbb{K}})$. We accordingly get that the mappings $\mathcal{S} : M \longmapsto \mathcal{S}(M)$ and $\Gamma : M \longmapsto \Gamma(M)$ define symmetric monoidal functors $\mathcal{S}, \Gamma : (\Sigma\mathrm{Seq}_{\mathbb{K}}, \otimes, \mathbb{K}) \longrightarrow (\mathrm{End}(\mathrm{Mod}_{\mathbb{K}}), \otimes, \mathbb{K})$.

— *We have natural isomorphisms*

$$\mathcal{S}(M \circ N) \simeq \mathcal{S}(M) \circ \mathcal{S}(N);$$

$$\Gamma(M \circ N) \simeq \Gamma(M) \circ \Gamma(N),$$

where $\circ$ denotes the composition of functors: $(F \circ G)(V) = F(G(V))$. We accordingly get that the mappings $\mathcal{S} : M \longmapsto \mathcal{S}(M)$ and $\Gamma : M \longmapsto \Gamma(M)$ define monoidal functors $\mathcal{S}, \Gamma : (\Sigma\mathrm{Seq}_{\mathbb{K}}, \tilde{\circ}, I) \longrightarrow (\mathrm{Mod}_{\mathbb{K}}, \circ, Id)$.

For every $M \in \Sigma\mathrm{Seq}_{\mathbb{K}}$, we have a natural transformation $Tr_M : \mathcal{S}(M,-) \longrightarrow \Gamma(M,-)$ defined, for every $V \in \mathrm{Mod}_{\mathbb{K}}$ and $v_1, \ldots, v_n \in V, m \in M(n)$, by

$$Tr_M(m \otimes v_1 \otimes \cdots \otimes v_n) = \sum_{\sigma \in \Sigma_n} (\sigma \cdot m) \otimes v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

This transformation is compatible with the above monoidal structures.

Remark A.2.14. *The natural transformation Tr is an isomorphism if $\mathbb{K}$ is of characteristic 0, but it is not an isomorphism in general.*

A.3 The notion of an operad

In this section, we introduce the notion of a (symmetric) operad, and everything that will be necessary in this thesis regarding this notion.

A.3.1 Definitions

The idea on the notion of an operad is to mimic a sequence of functions of several variables. This can be formalized as following.

Definition A.3.1. *A* (symmetric) operad *is a symmetric sequence $\mathcal{P}$ endowed with a distinguish element $1 \in \mathcal{P}(1)$ and morphisms*

$$\gamma : \mathcal{P}(r) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_r) \longrightarrow \mathcal{P}(n_1 + \cdots + n_r)$$

for every $r, n_1, \ldots, n_r \geq 0$, which fulfill the following axioms:

— *Associativity: for every* $p \in \mathcal{P}(n)$, $q_1 \in \mathcal{P}(s_1), \ldots, q_r \in \mathcal{P}(s_r)$, $\theta_i^j \in \mathcal{P}(n_i^j)$,

$$\begin{aligned}\gamma(\gamma(p \otimes q_1 \otimes \cdots \otimes q_r) \otimes \theta_1^1 \otimes \cdots \otimes \theta_1^{s_1} \otimes \cdots \otimes \theta_r^1 \otimes \cdots \otimes \theta_r^{s_r}) \\ = \gamma(p \otimes \gamma(q_1 \otimes \theta_1^1 \otimes \cdots \otimes \theta_1^{s_1}) \otimes \cdots \otimes \gamma(q_r \otimes \theta_r^1 \otimes \cdots \otimes \theta_r^{s_r})).\end{aligned}$$

— *Unit: for every* $p \in \mathcal{P}(r)$, $p(1, \ldots, 1) = p = 1(p)$.

— *Equivariance: for every* $p \in \mathcal{P}(r)$, $q_1 \in \mathcal{P}(s_1), \ldots, q_r \in \mathcal{P}(s_r), \sigma \in \Sigma_n, \tau_1 \in \Sigma_{s_1}, \ldots, \tau_r \in \Sigma_{s_r}$,

$$\gamma(\sigma \cdot p \otimes \tau_1 \cdot q_1 \otimes \cdots \otimes \tau_r \cdot q_r) = \sigma(\tau_1, \ldots, \tau_r) \cdot \gamma(p \otimes q_1 \otimes \cdots \otimes q_r).$$

A morphism of operads $\phi : \mathcal{P} \longrightarrow \mathcal{Q}$ *is a morphism of symmetric sequences which preserves the unit and the operadic composition.*

We denote by $\Sigma\mathcal{O}p$ the category of symmetric operads.

Remark A.3.2. *Let* $\mathcal{P} \in \Sigma\mathrm{Seq}_{\mathbb{K}}$. *Then giving the structure of an operad on* $\mathcal{P}$ *is equivalent to giving the structure of a monoid in the monoidal category* $(\Sigma\mathrm{Seq}_{\mathbb{K}}, \circledcirc, I)$.

We also have a notion of a non symmetric operad, which is the same definition as for a symmetric operad, but without the symmetric group actions and the equivariance axiom. We denote by $\mathcal{O}p$ the category of non symmetric operads. Note that we have a forgetful functor

$$\omega : \Sigma\mathcal{O}p \longrightarrow \mathcal{O}p$$

obtained by forgetting the group actions.

We give two important examples of operads. One example if the *operad of permutations*:

Proposition A.3.3. *The sequence* $(\mathbb{K}[\Sigma_n])_{n\geq 0}$ *is endowed with the structure of a symmetric operad. The* Σ_n *action on* $\mathbb{K}[\Sigma_n]$ *is given by left translations. The operadic composition is defined by*

$$\gamma(\sigma \otimes \tau_1 \otimes \cdots \otimes \tau_r) = \sigma(\tau_1, \ldots, \tau_r).$$

Another important example is the *endomorphism operad*:

Proposition A.3.4. *Let* $V \in \mathrm{Mod}_{\mathbb{K}}$. *For every* $n \geq 0$, *we set*

$$\mathrm{End}_V(n) = \mathrm{Mor}(V^{\otimes n}, V),$$

endowed with the Σ_n *action given by its action on* $V^{\otimes n}$. *Then* End_V *is a symmetric operad.*

Using the associativity and the unit axioms, we can build the full operadic composition γ from compositions with only two elements, which we call *partial compositions*.

Definition A.3.5. *We call* partial compositions *the operations* $\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \longrightarrow \mathcal{P}(n+m-1)$ *defined for every* $1 \leq i \leq n$ *by*

$$p \circ_i q = p(1, \ldots, \underset{i}{q}, \ldots, 1).$$

Remark A.3.6. *Using the associativity and unitality axioms, we see that the axioms for an operad can be expressed in terms of the partial compositions. It is even stronger: we can give an equivalent definition on the notion of an operad only in terms of partial compositions (see [LV12, §5.3.7] or [Fre17b, §2.1]).*

The category of symmetric operads is endowed with a monoidal structure given by the Hadamard tensor product.

Definition A.3.7. *Let $\mathcal{P}$ and $\mathcal{Q}$ be (symmetric) operads. We define the operad $\mathcal{P} \underset{\mathrm{H}}{\otimes} \mathcal{Q}$ by*

$$(\mathcal{P} \underset{\mathrm{H}}{\otimes} \mathcal{Q})(n) = \mathcal{P}(n) \otimes \mathcal{Q}(n)$$

endowed with the diagonal Σ_n action. The operadic compositions are given by

$$(\mathcal{P} \underset{H}{\otimes} \mathcal{Q})(n) \otimes (\mathcal{P} \underset{H}{\otimes} \mathcal{Q})(m) \xrightarrow{\simeq} \mathcal{P}(n) \otimes \mathcal{P}(m) \otimes \mathcal{Q}(n) \otimes \mathcal{Q}(m) \xrightarrow{\circ_i \otimes \circ_i} (\mathcal{P} \underset{H}{\otimes} \mathcal{Q})(n+m-1) \ .$$

A.3.2 The free operad functor

Theorem A.3.8. *Let M be a symmetric sequence and $\mathcal{P}$ be an operad. Then there exists a unique operad $\mathcal{F}(M)$, called the* free operad generated by *M, which comes with a symmetric sequence morphism $i : M \longrightarrow \mathcal{F}(M)$ such that, for every morphisms of symmetric sequences $f : M \longrightarrow \mathcal{P}$, there exists a unique morphism of operads $\phi_f : \mathcal{F}(M) \longrightarrow \mathcal{P}$ which makes the following diagram commutative:*

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "i"'] & & \mathcal{P} \\ & \mathcal{F}(M) \arrow[ur, dashed, "\exists ! \phi_f"'] & \end{tikzcd} .$$

There exist several equivalent ways to construct the operad $\mathcal{F}(M)$ associated to a symmetric sequence M. One way is recalled after Definition 2.6.2, using the language of trees.

A.3.3 Algebras over an operad

The main motivation for the notion of an operad is to generalize classical algebraic structures.

Definition A.3.9. *An* algebra over an operad *$\mathcal{P}$ (or $\mathcal{P}$-*algebra*) is a $\mathbb{K}$-module A endowed with morphisms*

$$\lambda : \mathcal{P}(r) \otimes A^{\otimes r} \longrightarrow A$$

which satisfy the following axioms.

- *Associativity: for every $p \in \mathcal{P}(r)$, $q_1 \in \mathcal{P}(s_1), \ldots, q_r \in \mathcal{P}(s_r), a_1^1, \ldots, a_1^{s_1}, \ldots, a_r^1 \ldots, a_r^{s_r} \in A$,*

$$\lambda(\gamma(p \otimes q_1 \otimes \cdots \otimes q_r) \otimes a_1^1 \otimes \cdots \otimes a_r^{s_r}) = \lambda(p \otimes \lambda(q_1 \otimes a_1^1 \otimes \cdots \otimes a_1^{s_1}) \otimes \cdots \otimes \lambda(q_r \otimes a_r^1 \otimes \cdots \otimes a_r^{s_r}));$$

- *Unit: for every $a \in A$, $\lambda(1 \otimes a) = a$;*

— *Equivariance: for every $p \in \mathcal{P}(r)$, $a_1, \ldots, a_r \in A$, $\sigma \in \Sigma_n$,*

$$\lambda(\sigma \cdot p \otimes a_1 \otimes \cdots \otimes a_r) = \lambda(p \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(r)}).$$

This notion can be characterized from an operadic point of view by the following proposition.

Proposition A.3.10. *Giving the structure of a $\mathcal{P}$-algebra on A is equivalent to giving a morphism of operads $\phi : \mathcal{P} \longrightarrow \mathrm{End}_A$.*

A.3.4 Monads $\mathcal{S}(\mathcal{P}, -)$ and $\Gamma(\mathcal{P}, -)$ associated to an operad $\mathcal{P}$

We first recall the notion of a monad.

Definition A.3.11. *Let $\mathcal{C}$ be a category. A* monad *is a functor $T : \mathcal{C} \longrightarrow \mathcal{C}$ endowed with a composition $\mu : T \circ T \longrightarrow T$ and a unit $\eta : id \longrightarrow T$ in the category of endofunctors of $\mathcal{C}$ such that the following diagrams commute:*

$$\begin{tikzcd} T \circ id \arrow[r, "id \circ \eta"] \arrow[dr, "="'] & T \circ T \arrow[d, "\mu"] & id \circ T \arrow[l, "\eta \circ id"'] \arrow[dl, "="] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} T \circ T \circ T \arrow[r, "id \circ \mu"] \arrow[d, "\mu \circ id"'] & T \circ T \arrow[d, "\mu"] \\ T \circ T \arrow[r, "\mu"'] & T \end{tikzcd}$$

A morphism of monads is a natural transformation $\phi : S \longrightarrow T$ which makes the following diagrams commutative:

$$\begin{tikzcd} & id \arrow[dl, "\eta_S"'] \arrow[dr, "\eta_T"] & \\ S \arrow[rr, "\phi"'] & & T \end{tikzcd} \qquad \begin{tikzcd} S \circ S \arrow[r, "\phi \circ \phi"] \arrow[d, "\mu_S"'] & T \circ T \arrow[d, "\mu_T"] \\ S \arrow[r, "\phi"'] & T \end{tikzcd}$$

Definition A.3.12. *Let (T, μ, η) be a monad in a category $\mathcal{C}$. An algebra over the monad (T, μ, η) is an object $A \in \mathcal{C}$ endowed with a morphism $\lambda : T(A) \longrightarrow A$ which makes the following diagram commutative:*

$$\begin{tikzcd} A \arrow[r, "\eta(A)"] \arrow[dr, "="'] & T(A) \arrow[d, "\lambda"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} T \circ T(A) \arrow[r, "T(\lambda)"] \arrow[d, "\mu(A)"'] & T(A) \arrow[d, "\lambda"] \\ T(A) \arrow[r, "\lambda"'] & A \end{tikzcd}$$

If A et B are algebras over the monad (T, μ, η), a morphism of algebras is a morphism $f : A \longrightarrow B$ which makes the following diagram commutative:

$$\begin{tikzcd} T(A) \arrow[r, "T(f)"] \arrow[d, "\lambda_A"'] & T(B) \arrow[d, "\lambda_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

In this thesis, we are mostly interested in two monads induced by an operad $\mathcal{P}$, for which we recall the construction.

Proposition A.3.13. *Let $\mathcal{P}$ be an operad. Then $\mathcal{P}$ is endowed with the structure of a monoid in the monoidal category $(\Sigma\mathrm{Seq}_{\mathbb{K}}, \circ, I)$. The monoid structure is given by the operadic composition $\gamma : \mathcal{P} \circ \mathcal{P} \longrightarrow \mathcal{P}$.*

We thus obtain the following corollary.

Corollary A.3.14. *Let $\mathcal{P}$ be an operad. Then the functor $\mathcal{S}(\mathcal{P}, -)$ is a monad whose algebras are exactly $\mathcal{P}$-algebras. The monad structure is given by the composite*

$$\mathcal{S}(\mathcal{P}, \mathcal{S}(\mathcal{P}, V)) \xrightarrow{\simeq} \mathcal{S}(\mathcal{P} \circ \mathcal{P}, V) \xrightarrow{\mathcal{S}(\gamma, id)} \mathcal{S}(\mathcal{P}, V) \ .$$

In general, the functor $\Gamma(\mathcal{P}, -)$ is not endowed with the structure of a monad, unless if $\mathcal{P}(0) = 0$.

Proposition A.3.15. *Let $\mathcal{P}$ be an operad such that $\mathcal{P}(0) = 0$. Then $\mathcal{P}$ is endowed with the structure of a monoid in the monoidal category $(\Sigma\mathrm{Seq}_{\mathbb{K}}, \tilde{\circ}, I)$. This monoid structure is defined by the composite*

$$\mathcal{P} \tilde{\circ} \mathcal{P} \xleftarrow[\simeq]{Tr_{\mathcal{P},\mathcal{P}}} \mathcal{P} \circ \mathcal{P} \xrightarrow{\gamma} \mathcal{P} \ .$$

Corollary A.3.16. *Let $\mathcal{P}$ be an operad such that $\mathcal{P}(0) = 0$. Then $\Gamma(\mathcal{P}, -)$ is a monad. The monad structure is given by the composite*

$$\Gamma(\mathcal{P}, \Gamma(\mathcal{P}, V)) \xrightarrow{\simeq} \Gamma(\mathcal{P} \tilde{\circ} \mathcal{P}, V) \xrightarrow{\Gamma(\gamma, id)} \Gamma(\mathcal{P}, V) \ .$$

Moreover, the trace map

$$Tr : \mathcal{S}(\mathcal{P}, -) \longrightarrow \Gamma(\mathcal{P}, -)$$

is a morphism of monads.

The algebras over this monad have an important role in this thesis.

Definition A.3.17. *Let $\mathcal{P}$ be an operad such that $\mathcal{P}(0) = 0$. A* $\Gamma\mathcal{P}$-algebra *(or $\mathcal{P}$-*algebra with divided powers*) is an algebra over the monad $\Gamma(\mathcal{P}, -)$.*

Remark A.3.18. *In particular, using the trace map, every $\Gamma\mathcal{P}$-algebra is a $\mathcal{P}$-algebra.*

The classical example is associated to commutative algebras. Let $\mathcal{C}om$ be the operad such that $\mathcal{C}om(0) = 0$ and $\mathcal{C}om(n) = \mathbb{K}$ for every $n \geq 1$ with trivial compositions and trivial symmetric groups actions. The operad $\mathcal{C}om$ governs associative and commutative algebras with no unit.

Example A.3.19. *Giving a $\Gamma(\mathcal{C}om, -)$-algebra is equivalent to giving a $\mathbb{K}$-module V endowed with operations $\gamma_n : V^{\otimes n} \longrightarrow V$ for every $n \geq 1$ which mimic the operations*

$$\gamma_n(x) = \frac{1}{n!} x^n.$$

Namely, the operations γ_n are required to satisfy the following formulas:

1. $\gamma_1 = id$;
2. $\gamma_n(x+y) = \gamma_n(x) + \sum_{\substack{p+q=n \\ p,q\neq 0}} \gamma_p(x)\gamma_q(y) + \gamma_n(y)$;
3. $\gamma_n(\lambda x) = \lambda^n \gamma_n(x)$;
4. $\gamma_n(x)\gamma_m(x) = \binom{n+m}{n}\gamma_{n+m}(x)$;
5. $\gamma_m(\gamma_n(x)) = \frac{(nm)!}{(n!)^m m!}\gamma_{nm}(x)$.

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